

Coupling by reflection of diffusion processes via discrete approximation under a backward Ricci flow

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Abstract

A coupling by reflection of a time-inhomogeneous diffusion process on a manifold are studied. The condition we assume is a natural time-inhomogeneous extension of lower Ricci curvature bounds. In particular, it includes the case of backward Ricci flow. As in time-homogeneous cases, our coupling provides a gradient estimate of the diffusion semigroup which yields the strong Feller property. To construct the coupling via discrete approximation, we establish the convergence in law of geodesic random walks as well as a uniform non-explosion type estimate.

1 INTRODUCTION

In stochastic analysis, coupling methods of stochastic processes have played a prominent role in the literature. Given two stochastic processes $Y_1(t)$ and $Y_2(t)$ on a state space M , a coupling $\mathbf{X}(t) = (X_1(t), X_2(t))$ of $Y_1(t)$ and $Y_2(t)$ is a stochastic process on $M \times M$ such that X_i has the same law as Y_i for $i = 1, 2$. By constructing a suitable coupling which reflects the geometry of the underlying structure, one can obtain various estimates for heat kernels, harmonic maps, eigenvalues etc. under natural geometric assumptions (see [11, 14, 27] for instance). Recently, the heat equation on time-inhomogeneous spaces such as Ricci flow have been studied intensively (see [1, 7, 18, 17, 19, 24, 29] and references therein). These studies have succeeded in revealing a tighter connection between the heat equation and the underlying geometric structure even in time-inhomogeneous cases. It should be remarked that an idea of coupling methods lies behind some of them [1, 17, 19, 24] in an essential way .

This paper is aimed at constructing a coupling by reflection of a diffusion process associated with a time-dependent family of metrics such as (backward) Ricci flow. Let M be a smooth manifold with a family of complete Riemannian metrics $\{g(t)\}_{t \in [T_1, T_2]}$. By $\{X(t)\}_{t \in [T_1, T_2]}$, we denote the $g(t)$ -Brownian motion. It means that $X(t)$ is the time-inhomogeneous diffusion process on M associated with $\Delta_{g(t)}/2$, where $\Delta_{g(t)}$ is the Laplacian with respect to $g(t)$ (see [7] for a construction of $g(t)$ -Brownian motion). As in

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time-homogeneous cases studied in [8, 13, 16, 25, 26, 27] under a lower Ricci curvature bound, a coupling by reflection $\mathbf{X}(t) = (X_1(t), X_2(t))$ of two $g(t)$ -Brownian motions starting from a different point provides us a useful control of the coupling time τ^* , the first time when coupled particles meet. A simple version of our main theorem which states such a control is as follows:

Theorem 1.1 *Suppose*

$$\partial_t g(t) \leq \text{Ric}_{g(t)} \quad (1.1)$$

holds. Then, for each $x_1, x_2 \in M$, there exists a coupling $\mathbf{X}(t) := (X_1(t), X_2(t))$ of two $g(t)$ -Brownian motions starting at (x_1, x_2) satisfying

$$\mathbb{P}[\tau^* > t] \leq \mathbb{P} \left[\inf_{T_1 \leq s \leq t} B(s) > -\frac{d_{g(T_1)}(x_1, x_2)}{2} \right] \quad (1.2)$$

for each t , where $d_{g(T_1)}$ is the distance function on M with respect to $g(T_1)$ and $B(t)$ is a 1-dimensional standard Brownian motion starting at the time T_1 .

For the complete statement of our main theorem, see Theorem 4.1. There we also study a diffusion process which generalizes the $g(t)$ -Brownian motion. The condition (1.1) can be interpreted as a time-inhomogeneous analogue of nonnegative Ricci curvature (see Remark 4.2). This condition is essentially the same as backward super Ricci flow in [19] (Our condition is slightly different in constant since our $g(t)$ -Brownian motion and hence the heat equation corresponds to $\Delta_{g(t)}/2$ instead of $\Delta_{g(t)}$). Obviously, (1.1) is satisfied if $g(t)$ evolves according to backward Ricci flow $\partial_t g(t) = \text{Ric}_{g(t)}$. As in the time-homogeneous case, our coupling time estimate yields a gradient estimate of the heat semigroup which implies the strong Feller property for the heat semigroup (see Corollary 4.3). Note that, when $g(t)$ is a backward Ricci flow, the same estimate as Corollary 4.3 is also obtained in [7] by using techniques in stochastic differential geometry.

To explain our approach to Theorem 1.1, let us review a heuristic idea of the construction of a coupling by reflection as well as that of the derivation of (1.2). Given a Brownian particle X_1 , we will construct X_2 by determining its infinitesimal motion $dX_2(t) \in T_{X_2(t)}M$ by using $dX_1(t) \in T_{X_1(t)}M$. First we take a minimal $g(t)$ -geodesic γ joining $X_1(t)$ and $X_2(t)$. Next, by using the parallel transport along γ associated with the $g(t)$ -Levi-Civita connection, we bring $dX_1(t)$ into $T_{X_2(t)}M$. Finally we define $dX_2(t)$ as a reflection of it with respect to a hyperplane being $g(t)$ -perpendicular to $\dot{\gamma}$ in $T_{X_2(t)}M$. From this construction, the Itô formula implies that $d_{g(t)}(X_1(t), X_2(t))$ should become a semimartingale at least until $(X_1(t), X_2(t))$ hits the $g(t)$ -cutlocus $\text{Cut}_{g(t)}$. The semimartingale decomposition is given by variational formulas of arclength. On the bounded variation part, there appear the time-derivative of $d_{g(t)}$ and the second variation of $d_{g(t)}$, which is dominated in terms of Ricci curvature. With the aid of our condition (1.1), these two terms are compensated and a nice domination of the bounded variation part follows. Thus the hitting time to 0 of $d_{g(t)}(X_1(t), X_2(t))$, which is the same as τ^* , can be estimated by that of the dominating semimartingale. Indeed, we can regard $2B(t) + d_{g(T_1)}(x_1, x_2)$ which appeared in the right hand side of (1.2) as the dominating semimartingale. The effect of our reflection appears in the martingale part $2B(t)$ which makes it possible for the dominating martingale to hit 0. This construction seems to work as long as $(X_1(t), X_2(t))$ is not in

the cutlocus. Moreover, we can hope it possible to construct it beyond the cutlocus since the cutlocus is sufficiently small and the effect of singularity at the cutlocus should make $d_{g(t)}(X_1(t), X_2(t))$ to decrease. If we succeed in doing so, the bounded variation part will involve a “local time at $\text{Cut}_{g(t)}$ ”. It will be negligible since it would be nonpositive. We can conclude that almost all technical difficulties are concentrated on the treatment of singularity at the cutlocus in order to make this heuristic argument rigorous. In fact, Theorem 1.1 is shown in [21] by using SDE methods when the $g(t)$ -cutlocus is empty for every $t \in [T_1, T_2]$.

Our construction of a coupling by reflection is based on a time-discretized approximation as studied in [16, 25]. We construct a coupling of geodesic random walks each of whose marginals approximates the original diffusion process. The construction will be finished after taking a limit so that these approximations converge. Our method has a remarkable advantage in treating singularities arising from the cutlocus. In our construction, we can avoid to extract a local time at $\text{Cut}_{g(t)}$ and directly obtain a dominating process which does not involve such a term. In the present framework, the singular set $\text{Cut}_{g(t)}$ also depends on time parameter t and hence treating it by using stochastic differential equations seems to be more complicated than in the time-homogeneous case.

Different kinds of couplings are studied in above-mentioned papers. Based on the theory of optimal transportation, McCann and Topping [19, 24] studied a coupling of heat distributions which minimizes their transportation cost. They used the squared distance in [19] or Perelman’s \mathcal{L} -functional in [24] respectively to quantify a transportation cost. Their coupling is closely related to coupling of Brownian motions by parallel transport along minimal (\mathcal{L} -)geodesics. In fact, studying a coupling by parallel transport by probabilistic methods recovered and extended (a part of) their results in [1] and [17] respectively. Note that our approach via time-discretized approximation is used in [17]. In addition, we also can construct a coupling by parallel transport by using our method to recover a result in [1] (see Theorem 4.6). It explains that our approach is also effective even when we study a different kind of couplings.

We give a remark on a difference in methods between ours and Arnaudon, Coulibaly and Thalmaier’s one [1] to construct a coupling by parallel transport. They consider one-parameter family of coupled particles along a curve. Intuitively saying, they concatenate coupled particles along a curve by iteration of making a coupling by parallel transport. Since “adjacent” particles are infinitesimally close to each other, we can ignore singularities on the cutlocus when we construct a coupled particle from an “adjacent” one. It should be noted that their method does not seem to be able to be applied directly in order to construct a coupling by reflection. Indeed, their construction of a chain of coupled particles heavily relies on a multiplicative (or semigroup) property of the parallel transport. However, our reflection operation obviously fails to possess such a multiplicative property. Since our reflection map changes orientation, there is no chance to interpolate it with a continuous family of isometries.

In what follows, we will state the organization of this paper. In the next section, we show basic properties of a family of Riemannian manifolds $((M, g(t)))_t$. In particular, we prove that Riemannian metrics $(g(t))_t$ are locally comparable with each other. It will be used to give a uniform control of several error terms which appear as a result of our discrete approximation. In section 3, we will study geodesic random walks in our time-

inhomogeneous framework. There we introduce them and prove the convergence in law to a diffusion process. After a small discussion at the beginning of the section, the proof is divided into two main parts. In the first part, we will give a uniform estimate for the exit time from a big compact set of geodesic random walks. Our assumption here is almost the same as in [18] where non-explosion of the diffusion process is studied (see Remark 3.3 (ii) for more details). In the second part, we prove tightness of geodesic random walks on the basis of the result in the first part. In section 4, we will construct a coupling by reflection and show an estimate of coupling time, which completes the proof of Theorem 1.1 as a special case.

2 PROPERTIES ON TIME-DEPENDENT METRIC

Let M be a m -dimensional manifold. For $-\infty < T_1 < T_2 < \infty$, Let $(g(t))_{t \in [T_1, T_2]}$ be a family of complete Riemannian metrics on M which smoothly depends on t .

Remark 2.1 *It seems to be restrictive that our time parameter only runs over the compact interval $[T_1, T_2]$. An example of $g(t)$ we have in mind is a solution to the backward Ricci flow equation. In this case, we can work on a semi-infinite interval $[T_1, \infty)$ only when we study an ancient solution of the Ricci flow. Thus $T_2 < \infty$ is not so restrictive. In addition, we could extend our results to the case on $[T_1, \infty)$ with a small modification of our arguments. It would be helpful to study an ancient solution. To deal with a singularity of Ricci flow, it could be nice to work on a semi-open interval $(T_1, T_2]$, where T_1 is the first time when a singularity emerges. In that case, we should be more careful since we cannot give “an initial condition at T_1 ” to define a $g(t)$ -Brownian motion on M .*

We collect some notations which will be used in the sequel. Throughout this paper, we fix a reference point $o \in M$. Let \mathbb{N}_0 be nonnegative integers. For $a, b \in \mathbb{R}$, $a \wedge b$ and $a \vee b$ stand for $\min\{a, b\}$ and $\max\{a, b\}$ respectively. Let $\text{Cut}_{g(t)}(x)$ be the set of the $g(t)$ -cutlocus of x on M . Similarly, the $g(t)$ -cutlocus $\text{Cut}_{g(t)}$ and the space-time cutlocus Cut_{ST} are defined by

$$\begin{aligned} \text{Cut}_{g(t)} &:= \{(x, y) \in M \times M \mid y \in \text{Cut}_{g(t)}(x)\}, \\ \text{Cut}_{\text{ST}} &:= \{(t, x, y) \in [T_1, T_2] \times M \times M \mid (x, y) \in \text{Cut}_{g(t)}\}. \end{aligned}$$

Set $D(M) := \{(x, x) \mid x \in M\}$. The distance function with respect to $g(t)$ is denoted by $d_{g(t)}(x, y)$. Note that Cut_{ST} is closed and that $d_{g(\cdot)}(\cdot, \cdot)$ is smooth on $[T_1, T_2] \times M \times M \setminus (\text{Cut}_{\text{ST}} \cup [T_1, T_2] \times D(M))$ (see [19], cf. [18]). We denote an open $g(s)$ -ball of radius R centered at $x \in M$ by $B_R^{(s)}(x)$. Some additional notations will be given at the beginning of the next section.

In the following three lemmas (Lemma 2.2-Lemma 2.4), we discuss a local comparison between $d_{g(t)}$ and $d_{g(s)}$ for $s \neq t$. Those will be a geometric basis of the further arguments.

Lemma 2.2 *Let M_0 be a compact subset of M . Then there exists $\kappa = \kappa(M_0)$ such that*

$$e^{-2\kappa|t-s|}g(s) \leq g(t) \leq e^{2\kappa|t-s|}g(s)$$

holds on M_0 for $t, s \in [T_1, T_2]$. In particular, if a minimal $g(s)$ -geodesic γ joining $x, y \in M_0$ is included in M_0 , then, for $t \in [T_1, T_2]$,

$$d_{g(t)}(x, y) \leq e^{\kappa|t-s|} d_{g(s)}(x, y).$$

Proof. Let $\pi : TM \rightarrow M$ be a canonical projection. Let us define \hat{M}_0 by

$$\hat{M}_0 := \left\{ (t, v) \in [T_1, T_2] \times TM \mid \pi(v) \in M_0, |v|_{g(t)} \leq 1 \right\}.$$

Note that \hat{M}_0 is closed since $g(\cdot)$ is continuous. We claim that \hat{M}_0 is sequentially compact. Let us take a sequence $((t_n, v_n))_{n \in \mathbb{N}} \subset \hat{M}_0$. We may assume $t_n \rightarrow t \in [T_1, T_2]$ and $\pi(v_n) \rightarrow p \in M_0$ as $n \rightarrow \infty$ by taking a subsequence if necessary. Let U be a neighborhood of p such that $\{v \in TM \mid \pi(v) \in U\} \simeq U \times \mathbb{R}^m$. For sufficiently large n , we regard v_n as an element of $U \times \mathbb{R}^m$ and write $v_n = (p_n, \tilde{v}_n)$. If we cannot take any convergent subsequence of $(v_n)_{n \in \mathbb{N}}$, then $|\tilde{v}_n| \rightarrow \infty$ as $n \rightarrow \infty$, where $|\cdot|$ stands for the standard Euclidean norm on \mathbb{R}^m (irrelevant to $(g(t))_{t \in [T_1, T_2]}$). Set $v'_n = (p_n, |\tilde{v}_n|^{-1} \tilde{v}_n)$. Then, there exists a subsequence $(v'_{n_k})_{k \in \mathbb{N}} \subset (v'_n)_{n \in \mathbb{N}}$ such that $v'_{n_k} \rightarrow v'_\infty = (p, \bar{v}')$ as $n \rightarrow \infty$ for some $\bar{v}' \in \mathbb{R}^m$ with $|\bar{v}'| = 1$. Since $g(\cdot)$ is continuous, $g(t_{n_k})(v'_{n_k}, v'_{n_k}) \rightarrow g(t)(v'_\infty, v'_\infty)$ as $k \rightarrow \infty$. On the other hand, $g(t_{n_k})(v'_{n_k}, v'_{n_k}) \leq |\tilde{v}_{n_k}|^{-2} \rightarrow 0$ since $g(t_n)(v_n, v_n) \leq 1$. Thus \bar{v}' must be 0. It contradicts with $|\bar{v}'| = 1$. Hence \hat{M}_0 is sequentially compact.

Since $\hat{M}_0 \ni (t, v) \mapsto \partial_t g(t)(v, v)$ is continuous, there exists a constant $\kappa = \kappa(M_0) > 0$ such that $|\partial_t g(t)(v, v)| \leq 2\kappa$ for every $(t, v) \in \hat{M}_0$. Take $v \in \pi^{-1}(M_0)$, $v \neq 0_{\pi(v)}$. Then

$$\partial_t g(t)(v, v) = |v|_{g(t)}^2 \partial_t g(t)(|v|_{g(t)}^{-1} v, |v|_{g(t)}^{-1} v) \leq 2\kappa |v|_{g(t)}^2.$$

Thus $\partial_t \log g(t)(v, v) \leq 2\kappa$ holds. By integrating it from s to t with $s < t$, we obtain $g(t)(v, v) \leq e^{2\kappa(t-s)} g(s)(v, v)$. We can obtain the other inequality similarly.

For the latter assertion, for a, b with $\gamma(a) = x$ and $\gamma(b) = y$,

$$d_{g(t)}(x, y) \leq \int_a^b |\dot{\gamma}(u)|_{g(t)} du \leq e^{\kappa|t-s|} \int_a^b |\dot{\gamma}(u)|_{g(s)} du = e^{\kappa|t-s|} d_{g(s)}(x, y).$$

□

Lemma 2.3 For $R > 0$, $x \in M$ and $t \in [T_1, T_2]$, there exists $\delta = \delta(x, t, R) > 0$ such that $\bar{B}_r^{(s)}(x) \subset \bar{B}_{3r}^{(t)}(x)$ for $r \leq R$ and $s \in [T_1, T_2]$ with $|s - t| \leq \delta$.

Proof. Set $\kappa := \kappa(\bar{B}_{3R}^{(t)}(x))$ as in Lemma 2.2 and $\delta := \kappa^{-1} \log 2$. Take $p \in \bar{B}_r^{(s)}(x)$ and a minimal $g(s)$ -geodesic $\gamma : [a, b] \rightarrow M$ joining x and p . Suppose that there exists $u_0 \in [a, b]$ such that $\gamma(u_0) \in \bar{B}_{3r}^{(t)}(x)^c$. Let $\bar{u}_0 := \inf\{u \in [a, b] \mid \gamma(u) \in \bar{B}_{3r}^{(t)}(x)^c\}$. Since $\gamma([a, \bar{u}_0]) \subset \bar{B}_{3r}^{(t)}(x) \subset \bar{B}_{3R}^{(t)}(x)$ and $d_{g(t)}(x, \gamma(\bar{u}_0)) = 3r$, Lemma 2.2 yields

$$d_{g(s)}(x, p) \geq \int_a^{\bar{u}_0} |\dot{\gamma}(u)|_{g(s)} du \geq e^{-\kappa\delta} \int_a^{\bar{u}_0} |\dot{\gamma}(u)|_{g(t)} du = \frac{3r}{2}.$$

This is absurd. Hence $\gamma([a, b]) \subset \bar{B}_{3r}^{(t)}(x)$. In particular, $\gamma(b) = p \in \bar{B}_{3r}^{(t)}(x)$. □

Lemma 2.4 *For $R > 0$, there exists a compact subset $M_0 = M_0(R)$ of M such that*

$$\left\{ p \in M \mid \inf_{t \in [T_1, T_2]} d_{g(t)}(o, p) \leq R \right\} \subset M_0. \quad (2.1)$$

Proof. For each $t \in [T_1, T_2]$, take $\delta(o, t, R + 1) > 0$ according to Lemma 2.3. Take $\{t_i\}_{i=1}^n \subset [T_1, T_2]$ such that

$$[T_1, T_2] \subset \bigcup_{i=1}^n (t_i - \delta(o, t_i, R + 1), t_i + \delta(o, t_i, R + 1)).$$

Let us define a compact set $M_0 \subset M$ by $M_0 := \bigcup_{i=1}^n \bar{B}_{3R}^{(t_i)}(o)$. Take $p \in M$ such that $\inf_{T_1 \leq t \leq T_2} d_{g(t)}(o, p) \leq R$. For $\varepsilon \in (0, 1)$, take $s \in [T_1, T_2]$ such that $d_{g(s)}(o, p) \leq R + \varepsilon$. Then there exists $j \in \{1, \dots, n\}$ such that $|s - t_j| < \delta(o, t_j, R + 1)$. By Lemma 2.3, it implies $p \in \bar{B}_{R+\varepsilon}^{(s)}(o) \subset \bar{B}_{3(R+\varepsilon)}^{(t_j)}(o) \subset \bigcup_{i=1}^n \bar{B}_{3(R+\varepsilon)}^{(t_i)}(o)$. Hence the conclusion follows by letting $\varepsilon \downarrow 0$. \square

Another useful consequence of Lemma 2.2 and Lemma 2.3 is the following:

Lemma 2.5 *$d_{g(\cdot)}(\cdot, \cdot)$ is continuous on $[T_1, T_2] \times M \times M$.*

Proof. Since the topology on $[T_1, T_2] \times M \times M$ is metrizable, It suffices to show $\lim_{n \rightarrow \infty} d_{g(t_n)}(x_n, y_n) = d_{g(t)}(x, y)$ when $(t_n, x_n, y_n) \rightarrow (t, x, y)$ as $n \rightarrow \infty$. By the triangle inequality,

$$|d_{g(t_n)}(x_n, y_n) - d_{g(t)}(x, y)| \leq |d_{g(t_n)}(x, y) - d_{g(t)}(x, y)| + d_{g(t_n)}(x, x_n) + d_{g(t_n)}(y, y_n). \quad (2.2)$$

Take $R > 0$ so that $B_R^{(t)}(x)$ includes a minimal $g(t)$ -geodesic joining x and y . Take $\kappa = \kappa(\bar{B}_{4R}^{(t)}(x))$ according to Lemma 2.2. We can easily see that every minimal $g(t)$ -geodesic joining y and y_n is included in $B_{2R}^{(t)}(x)$ for sufficiently large $n \in \mathbb{N}$. Thus Lemma 2.2 yields

$$\limsup_{n \rightarrow \infty} d_{g(t_n)}(y, y_n) \leq \limsup_{n \rightarrow \infty} e^{\kappa|t-t_n|} d_{g(t)}(y, y_n) = 0.$$

We can show $d_{g(t_n)}(x, x_n) \rightarrow 0$ similarly. Take a minimal $g(t_n)$ -geodesic $\gamma_n : [a, b] \rightarrow M$ joining x and y . By our choice of R , Lemma 2.2 again yields

$$d_{g(t_n)}(x, \gamma_n(u)) \leq d_{g(t_n)}(x, y) \leq e^{\kappa|t-t_n|} d_{g(t)}(x, y) \leq e^{\kappa|t-t_n|} R.$$

It implies $\limsup_{n \rightarrow \infty} d_{g(t_n)}(x, y) \leq d_{g(t)}(x, y)$. In addition, γ_n is included in $B_{4R/3}^{(t_n)}(x)$ for sufficiently large n . Thus Lemma 2.3 and Lemma 2.2 yield $d_{g(t)}(x, y) \leq e^{\kappa|t-t_n|} d_{g(t_n)}(x, y)$. Hence the conclusion follows by combining these estimates with (2.2). \square

Before closing this section, we will provide a local lower bound of injectivity radius which is uniform in time parameter.

Lemma 2.6 *For every $M_1 \subset M$ compact, there is $\tilde{r}_0 = \tilde{r}_0(M_1) > 0$ such that $d_{g(t)}(y, z) < \tilde{r}_0$ implies $(t, y, z) \notin \text{Cut}_{\text{ST}}$ for any $(t, y, z) \in [T_1, T_2] \times M_1 \times M_1$.*

Proof. Take $R > 1$ so that $\sup_{t \in [T_1, T_2]} \sup_{x \in M_1} d_{g(t)}(o, x) < R - 1$. By Lemma 2.4, there exists a compact set $M_0 \subset M$ such that (2.1) holds. For every $t \in [T_1, T_2]$ and $x \in M_1$, $(t, x, x) \notin \text{Cut}_{\text{ST}}$. It implies that there is $\eta_{t,x} \in (0, 1)$ such that $(s, y, z) \notin \text{Cut}_{\text{ST}}$ whenever $d_{g(t)}(x, y) \vee d_{g(t)}(x, z) \vee |t - s| < \eta_{t,x}$ since Cut_{ST} is closed. Thus there exist $N \in \mathbb{N}$ and $(t_i, x_i) \in [T_1, T_2] \times M_1$ ($i = 1, \dots, N$) such that

$$[T_1, T_2] \times M_1 \subset \bigcup_{i=1}^N \left(t_i - \frac{\eta_{t_i, x_i}}{2}, t_i + \frac{\eta_{t_i, x_i}}{2} \right) \times B_{\eta_{t_i, x_i}/2}^{(t_i)}(x_i).$$

Set $\tilde{r}_0 > 0$ by

$$\tilde{r}_0 := \frac{1}{2} \exp \left(-\frac{\kappa}{2} \max_{1 \leq i \leq N} \eta_{t_i, x_i} \right) \min_{1 \leq i \leq N} \eta_{t_i, x_i},$$

where $\kappa = \kappa(M_0) > 0$ is as in Lemma 2.2. Take $(s, y, z) \in [T_1, T_2] \times M_1 \times M_1$ with $d_{g(s)}(y, z) < \tilde{r}_0$. Take $j \in \{1, \dots, N\}$ so that $|s - t_j| \vee d_{g(t_j)}(x_j, y) < \eta_{t_j, x_j}/2$. By virtue of the choice of R and M_0 , Lemma 2.4 yields that every $g(s)$ -geodesic joining y and z is included in M_0 . Thus Lemma 2.2 yields

$$d_{g(t_j)}(y, z) \leq e^{\kappa|s - t_j|} d_{g(s)}(y, z) < \frac{\eta_{t_j, x_j}}{2}.$$

It implies $|s - t_j| \vee d_{g(t_j)}(x_j, y) \vee d_{g(t_j)}(x_j, z) < \eta_{t_j, x_j}$ and hence $(s, y, z) \notin \text{Cut}_{\text{ST}}$. \square

3 APPROXIMATION VIA GEODESIC RANDOM WALKS

Let $(Z(t))_{t \in [T_1, T_2]}$ be a family of smooth vector fields continuously depending on the parameter $t \in [T_1, T_2]$. Let $X(t)$ be the diffusion process associated with the time-dependent generator $\mathcal{L}_t = \Delta_{g(t)}/2 + Z(t)$ (see [7] for a construction of $X(t)$ by solving a SDE on the frame bundle). Note that $(t, X(t))$ is a unique solution to the martingale problem associated with $\partial_t + \mathcal{L}$ on $[T_1, T_2] \times M$ (see [11] for the time-homogeneous case. Its extension to time-inhomogeneous case is straightforward; see [23] also).

In what follows, we will use several notions in Riemannian geometry such as exponential map \exp , Levi-Civita connection ∇ , Ricci curvature Ric etc. To clarify the dependency on the metric $g(t)$, we put (t) on superscript or $g(t)$ on subscript. For instance, we use the following symbols: $\exp^{(t)}$, $\nabla^{(t)}$ and $\text{Ric}_{g(t)}$. We refer to [6] for basics in Riemannian geometry which will be used in this paper.

For each $t \in [T_1, T_2]$, we fix a measurable section $\Phi^{(t)} : M \rightarrow \mathcal{O}^{(t)}(M)$ of the $g(t)$ -orthonormal frame bundle $\mathcal{O}^{(t)}(M)$ of M . Take a sequence of independent, identically distributed random variables $\{\xi_n\}_{n \in \mathbb{N}}$ which are uniformly distributed on the unit disk in \mathbb{R}^m . Given $x_0 \in M$, let us define a continuously-interpolated geodesic random walk $(X^\alpha(t))_{t \in [T_1, T_2]}$ on M starting from x_0 with a scale parameter $\alpha > 0$ inductively. Let $t_n^{(\alpha)} := (T_1 + \alpha^2 n) \wedge T_2$ for $n \in \mathbb{N}_0$. For $t = T_1 = t_0^{(\alpha)}$, set $X^\alpha(T_1) := x_0$. after $X^\alpha(t)$ is defined for $t \in [T_1, t_n^{(\alpha)}]$, we extend it to $t \in [t_n^{(\alpha)}, t_{n+1}^{(\alpha)}]$ by

$$\begin{aligned} \tilde{\xi}_{n+1} &:= \sqrt{m+2} \Phi^{(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) \xi_{n+1}, \\ X^\alpha(t) &:= \exp_{X^\alpha(t_n^{(\alpha)})}^{(t_n^{(\alpha)})} \left(\frac{t - t_n^{(\alpha)}}{\alpha^2} \left(\alpha \tilde{\xi}_{n+1} + \alpha^2 Z(t_n^{(\alpha)}) \right) \right). \end{aligned}$$

For later use, we define $N^{(\alpha)} := \inf \{n \in \mathbb{N}_0 \mid t_{n+1} - t_n < \alpha^2\}$. This is the total number of discrete steps of our geodesic random walks with scale parameter α . Set $\mathcal{C} := C([T_1, T_2] \rightarrow M)$ and $\mathcal{D} := D([T_1, T_2] \rightarrow M)$. By using a distance $d_{g(T_1)}$ on M , we metrize \mathcal{C} and \mathcal{D} as usual so that \mathcal{C} and \mathcal{D} become Polish spaces (see [9] for a distance function on \mathcal{D} , for example). Set $\mathcal{C}_1 := C([T_1, T_2] \rightarrow [0, \infty))$. Let us define a time-dependent $(0, 2)$ -tensor field $(\nabla Z(t))^\flat$ by

$$(\nabla Z(t))^\flat(X, Y) := \frac{1}{2} \left(\langle \nabla_X^{(t)} Z(t), X \rangle_{g(t)} + \langle \nabla_Y^{(t)} Z(t), Y \rangle_{g(t)} \right).$$

Assumption 1 *There exists a locally bounded nonnegative measurable function b on $[0, \infty)$ such that*

(i) *For all $t \in [T_1, T_2]$,*

$$2(\nabla Z(t))^\flat + \partial_t g(t) \leq \text{Ric}_{g(t)} + b(d_{g(t)}(o, \cdot))g(t).$$

(ii) *For each $C > 0$, a 1-dimensional diffusion process y_t given by*

$$dy_t = d\beta_t + \frac{1}{2} \left(C + \int_0^{y_t} b(s) ds \right) dt$$

does not explode. (This is the case if and only if

$$\int_1^\infty \exp \left(- \int_1^y \mathbf{b}(z) dz \right) \int_1^y \exp \left(\int_1^z \mathbf{b}(\xi) d\xi \right) dz dy = \infty,$$

where $\mathbf{b}(y) := C + \int_0^y b(s) ds$. see e.g. [12, Theorem VI.3.2].)

Our goal in this section is to prove the following:

Theorem 3.1 *Under Assumption 1, X^α converges in law to X in \mathcal{C} as $\alpha \rightarrow 0$.*

Most of arguments in this section will be devoted to show the tightness i.e.

Proposition 3.2 *$(X^\alpha)_{\alpha \in (0,1)}$ is tight in \mathcal{C} .*

In fact, as we will see in the following, Proposition 3.2 easily implies Theorem 3.1.

Proof of Theorem 3.1. By virtue of Proposition 3.2, for any subsequence of $(X^\alpha)_{\alpha \in (0,1)}$ there exists a further subsequence $(X^{\alpha_k})_{k \in \mathbb{N}}$ which converges in law in \mathcal{C} as $k \rightarrow \infty$. Thus it suffices to show that this limit has the same law as X . Let $(\beta^\alpha(t))_{t \in [0, \infty)}$ be a Poisson process of intensity α^{-2} which is independent of $\{\xi_n\}_{n \in \mathbb{N}}$. Set

$$\bar{\beta}^\alpha(t) := (T_1 + \alpha^2 \beta^\alpha(t - T_1)) \wedge t_{N^{(\alpha)}}^{(\alpha)}.$$

Then the Poisson subordination $X^{\alpha_k}(\bar{\beta}^{\alpha_k}(\cdot))$ also converges in law in \mathcal{D} to the same limit (see [4] for instance). Note that $(\bar{\beta}^\alpha(t), X^\alpha(\bar{\beta}^\alpha(t)))_{t \in [T_1, T_2]}$ is a space-time Markov process. The associated semigroup $P_t^{(\alpha)}$ and its generator $\tilde{\mathcal{L}}^{(\alpha)}$ are given by

$$P_t^{(\alpha)} f := e^{-(t-T_1)\alpha^{-2}} \left(\sum_{l=1}^{N^{(\alpha)}} \frac{((t-T_1)\alpha^{-2})^l}{l!} (q^{(\alpha)})^l f + \sum_{l > N^{(\alpha)}} \frac{((t-T_1)\alpha^{-2})^l}{l!} (q^{(\alpha)})^{N^{(\alpha)}} f \right),$$

$$\tilde{\mathcal{L}}^{(\alpha)} f := \alpha^{-2} (q^{(\alpha)} f - f),$$

where

$$q^{(\alpha)} f(t, x) := \mathbb{E} \left[f(t + \alpha^2, \exp_x^{(t)} \left(\alpha \sqrt{m+2} \Phi^{(t)}(x) \xi_1 + \alpha^2 Z(t) \right) \right].$$

We can easily prove $\tilde{\mathcal{L}}^{(\alpha)} f \rightarrow (\partial_t + \mathcal{L})f$ uniformly as $\alpha \rightarrow 0$ for $f \in C_0^\infty([T_1, T_2] \times M)$. Since $(\bar{\beta}^\alpha(t), X^\alpha(\bar{\beta}^\alpha(t)))_{t \in [T_1, T_2]}$ is a solution to the martingale problem associated with $\tilde{\mathcal{L}}^{(\alpha)}$, the limit in law of $(\bar{\beta}^{\alpha_k}(t), X^{\alpha_k}(\bar{\beta}^{\alpha_k}(t)))_{t \in [T_1, T_2]}$ solves the martingale problem associated with $\partial_t + \mathcal{L}$. By the uniqueness of the martingale problem, this limit has the same law as that of $(t, X(t))_{t \in [T_1, T_2]}$. It completes the proof. \square

Remark 3.3 (i) A result on a convergence of semigroups [15] was used to show the convergence of finite dimensional distributions in the time-homogeneous case [5] (see [25] also). It is not so clear that we can employ the same argument in our time-inhomogeneous case. One difficulty arises from the absence of invariant measures for semigroups even in the case $Z(t) \equiv 0$. Although the $g(t)$ -Riemannian measure is a unique invariant measure for $\Delta_{g(t)}$, this measure also depends on time parameter. Thus we cannot expect that it becomes an invariant measure of semigroups. This obstacle also prevents us to employ the existing theory of time-dependent Dirichlet forms (see [20] for instance) in order to study our problem.

(ii) Proposition 3.2 also asserts that any subsequential limit in law is a probability measure on \mathcal{C} . Since we have not added any cemetery point to M in the definition of \mathcal{C} , Theorem 3.1 implies that X cannot explode. It almost recovers the result in [18]. Our assumption is slightly stronger than that in [18] on the point where we require (ii) for all $C > 0$, not a given constant. Note that we will use Assumption 1 (ii) only for a specified constant $2C_0$ given in Lemma 3.9. However, its expression looks complicated and it seems to be less interesting to provide a explicit bound.

Now we introduce some additional notations which will be used in the rest of this paper. For $t \in [T_1, T_2]$, we define $[t]_\alpha$ by $[t]_\alpha := \sup \{\alpha^2 n + T_1 \mid n \in \mathbb{N}_0, \alpha^2 n + T_1 < t\}$. Set $\mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$. For $R > 1$, let us define $\sigma_R : \mathcal{C}_1 \rightarrow [T_1, T_2] \cup \{\infty\}$ by

$$\sigma_R(w) := \inf \{t \in [T_1, T_2] \mid w(t) > R - 1\},$$

where $\inf \emptyset = \infty$. We write $\hat{\sigma}_R := \sigma_R(d_{g(\cdot)}(o, X^\alpha(\cdot)))$ and $\bar{\sigma}_R := \alpha^{-2}([\hat{\sigma}_R]_\alpha - T_1) + 1$. Note that $\bar{\sigma}_R$ is an \mathcal{F}_n -stopping time. For each $t \in [T_1, T_2]$ and $x, y \in M$ with $x \neq y$, we choose a minimal unit-speed $g(t)$ -geodesic $\gamma_{xy}^{(t)} : [0, d_{g(t)}(x, y)] \rightarrow M$ from x to y . Note that we can choose $\gamma_{xy}^{(t)}$ so that $(x, y) \mapsto \gamma_{xy}^{(t)}$ is measurable in an appropriate sense (see e.g. [25]). We use the same symbol $\gamma_{xy}^{(t)}$ for its range $\gamma_{xy}^{(t)}([0, d_{g(t)}(x, y)])$.

3.1 UNIFORM BOUND FOR ESCAPE PROBABILITY

The goal of this subsection is to show the following:

Proposition 3.4 $\lim_{R \uparrow \infty} \limsup_{\alpha \downarrow 0} \mathbb{P}[\hat{\sigma}_R \leq T_2] = 0.$

For the proof, we will establish a discrete analogue of a comparison argument for the radial process as discussed in [18]. In this subsection, we fix $R > 1$ sufficiently large so that $d_{g(T_1)}(o, x_0) < R - 1$ until the final line of the proof of Proposition 3.4. We also fix a relatively compact open set $M_0 \subset M$ satisfying (2.1). Set $r_0 := \tilde{r}_0 \wedge (1/2)$, where $\tilde{r}_0 = \tilde{r}_0(M_0)$ is as in Lemma 2.6.

The first step for proving Proposition 3.4 is to show a difference inequality for the radial process $d_{g(t)}(o, X^\alpha(t))$ (Lemma 3.7). It will play a role of the Itô formula for the radial process in our discrete setting. We introduce some notations to discuss how to avoid the singularity of $d_{g(\cdot)}(o, \cdot)$ on $\{o\} \cup \text{Cut}_{g(\cdot)}(o)$. For $r > 0$, let us define a set A'_r, A''_r and A_r as follows:

$$\begin{aligned} A'_r &:= \left\{ (t, x, y) \in [T_1, T_2] \times M_0 \times M_0 \mid \begin{array}{l} d_{g(t)}(x, x') + d_{g(t)}(y, y') + |t - t'| \geq r \\ \text{for any } (t', x', y') \in \text{Cut}_{\text{ST}} \end{array} \right\}, \\ A''_r &:= \left\{ (t, x, y) \in [T_1, T_2] \times M_0 \times M_0 \mid d_{g(t)}(x, y) \geq r \right\}, \\ A_r &:= A'_r \cap A''_r. \end{aligned}$$

Note that A_r is compact and that $d_{g(\cdot)}(\cdot, \cdot)$ is smooth on A_r . For $t \in [T_1, T_2]$ and $p \in M$, let us define $o_p^{(t)} \in M_0$ by

$$o_p^{(t)} := \begin{cases} \gamma_{op}^{(t)}\left(\frac{r_0}{2}\right) & \text{if } (t, o, p) \notin A'_{r_0}, \\ o & \text{otherwise.} \end{cases}$$

For simplicity of notations, we denote $o_{X^\alpha(t_n^{(\alpha)})}^{(t_n^{(\alpha)})}$ by o_n . Similarly, we use the symbol γ_n for $\gamma_{o_n X^\alpha(t_n^{(\alpha)})}^{(t_n^{(\alpha)})}$ throughout this section. Note that $(t, o_p^{(t)}, p) \notin \text{Cut}_{\text{ST}}$ holds. Furthermore, it is uniformly separated from Cut_{ST} in the following sense:

Lemma 3.5 *There exist $r_1 > 0$ and $\delta_1 > 0$ such that the following holds: Let $t_0, t \in [T_1, T_2]$ with $t - t_0 \in [0, \delta_1]$. Let $p_0 \in B_{R-1}^{(t_0)}(o)$ and $p \in B_{\delta_1}^{(t_0)}(p_0)$. Then we have*

- (i) $d_{g(t)}(o, p) \leq e^{\kappa(t-t_0)} (d_{g(t_0)}(o, p_0) + d_{g(t_0)}(p_0, p)),$
- (ii) $(t, o_{p_0}^{(t_0)}, p) \in A_{r_1}$ when $p_0 \notin B_{r_0}^{(t_0)}(o).$

Here $\kappa = \kappa(M_0) > 0$ is given according to Lemma 2.2.

By applying Lemma 3.5 to X^α , we obtain the following:

Corollary 3.6 *There exist $\alpha_0 > 0$ and $h : [0, \alpha_0] \rightarrow [0, 1]$ with $\lim_{\alpha \downarrow 0} h(\alpha) = 0$ such that the following holds: For $\alpha \leq \alpha_0$, $n \in \mathbb{N}_0$ and $s, t \in [t_n^{(\alpha)}, t_{n+1}^{(\alpha)}]$, when $n < \bar{\sigma}_R$,*

- (i) $d_{g(t)}(o, X^\alpha(s)) \leq e^{\kappa\alpha^2} (d_{g(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)})) + h(\alpha)),$

(ii) $(t, o_n, X^\alpha(s)) \in A_{r_1}$ when $X^\alpha(t_n^{(\alpha)}) \notin B_{r_0^{(t_n^{(\alpha)})}}(o)$.

Here r_1 is the same as in Lemma 3.5.

Proof of Corollary 3.6. Set $\bar{Z} := \sup_{t \in [T_1, T_2], x \in M_0} |Z(t)|_{g(t)}(x)$. Note that we have $d_{g(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)}), X^\alpha(t)) \leq \sqrt{m+2}\alpha + \bar{Z}\alpha^2$ by the definition of X^α . Take $\alpha_0 > 0$ so that $\sqrt{m+2}\alpha_0 + \bar{Z}\alpha_0^2 \leq \delta_1$ and $\alpha^2 \leq \delta_1$ hold, where δ_1 is as in Lemma 3.5. Then the conclusion follows by applying Lemma 3.5 with $t_0 = t_n^{(\alpha)}$, $p_0 = X^\alpha(t_n^{(\alpha)})$ and $p = X^\alpha(s)$. \square

Proof of Lemma 3.5. We show that (i) holds with $\delta_1 = 1$. By the triangle inequality, the proof is reduced to showing the following two inequalities:

$$d_{g(t)}(o, p_0) \leq e^{\kappa(t-t_0)} d_{g(t_0)}(o, p_0), \quad (3.1)$$

$$d_{g(t)}(p_0, p) \leq e^{\kappa(t-t_0)} d_{g(t_0)}(p_0, p). \quad (3.2)$$

Our condition (2.1) yields that $\gamma_{op_0}^{(t_0)}$ is included in M_0 . Thus Lemma 2.2 yields (3.1). When $p \in B_1^{(t_0)}(p_0)$, we have $p \in B_R^{(t_0)}(o)$. Hence (2.1) and Lemma 2.2 yield (3.2) in a similar way as (3.1).

Let us turn to consider (ii). For simplicity of notations, we denote $o_{p_0}^{(t_0)}$ by o' in this proof. We assume that $t - t_0 \in [0, \delta]$ and $p \in B_\delta^{(t_0)}(p_0)$ hold for $\delta > 0$. First we will show $(t, o', p) \in A''_{r_0/4}$ when δ is sufficiently small. Note that $(t_0, o', p_0) \in A''_{r_0/2}$ holds since $p_0 \notin B_{r_0}^{(t_0)}(o)$ and $d_{g(t_0)}(o, o') \in \{r_0/2, 0\}$. Let $q \in \gamma_{o'p_0}^{(t)}$. By the triangle inequality,

$$d_{g(t)}(o, q) \leq d_{g(t)}(o, o') + d_{g(t)}(o', p_0). \quad (3.3)$$

Since $r_0/2 < 1 < R$ holds, (2.1) yields $\gamma_{oo'}^{(t_0)} \subset M_0$ when $o' \neq o$. We can easily see that $\gamma_{o'p_0}^{(t_0)} \subset \gamma_{op_0}^{(t_0)} \subset M_0$. Thus, by applying Lemma 2.2 to (3.3),

$$\begin{aligned} d_{g(t)}(o, q) &\leq e^{\kappa(t-t_0)} (d_{g(t_0)}(o, o') + d_{g(t_0)}(o', p_0)) \\ &= (R-1)e^{\kappa\alpha^2}. \end{aligned} \quad (3.4)$$

Take $\delta_2 := 1 \wedge (\kappa^{-1} \log(R/(R-1)))$. Then, for any $\delta \in (0, \delta_2)$, (3.4) and (2.1) imply $\gamma_{o'p_0}^{(t)} \subset M_0$. Hence the triangle inequality, Lemma 2.2 and (3.2) yield

$$\begin{aligned} d_{g(t)}(o', p) &\geq d_{g(t)}(o', p_0) - d_{g(t)}(p_0, p) \\ &\geq e^{-\kappa(t-t_0)} d_{g(t_0)}(o', p_0) - e^{\kappa(t-t_0)} d_{g(t_0)}(p_0, p) \\ &\geq \frac{e^{-\kappa\delta} r_0}{2} - e^{\kappa\delta} \delta \end{aligned} \quad (3.5)$$

when $\delta \leq \delta_2$. Thus there exists $\delta_3 = \delta_3(\kappa, r_0, R) \in (0, \delta_2]$ such that the right hand side of (3.5) is greater than $r_0/4$ whenever $\delta \in (0, \delta_3)$. Hence $(t, o', p) \in A''_{r_0/4}$ holds in such a case.

Next we will show that there exists $r'_1 > 0$ such that $(t, o', p) \in A'_{r'_1}$ holds for sufficiently small δ . Once we have shown it, the conclusion holds with $r_1 = r'_1 \wedge (r_0/4)$. As we did

in showing $(t, o', p) \in A''_{r_0/4}$, we begin with studying the corresponding statement for (t_0, o', p_0) . More precisely, we claim that there exists $r''_1 \in (0, 1)$ such that $(t_0, o', p_0) \in A'_{r''_1}$ for all $\delta \in (0, 1)$. When $o' = o$, $(t_0, o', p_0) \in A'_{r_0}$ directly follows from the definition of $o' = o_{p_0}^{(t_0)}$. When $o' \neq o$, set

$$H := \left\{ (t, x, y) \in [T_1, T_2] \times \bar{M}_0 \times \bar{M}_0 \mid \begin{array}{l} r_0 \leq d_{g(t)}(o, y) \leq R - 1, \ d_{g(t)}(o, x) = r_0/2, \\ d_{g(t)}(x, y) = d_{g(t)}(o, y) - d_{g(t)}(o, x) \end{array} \right\}.$$

Note that H is compact and that $H \cap \text{Cut}_{\text{ST}} = \emptyset$ holds since $(t, x, y) \in H$ implies that x is on a minimal $g(t)$ -geodesic from y to o . Since $(t_0, o', p_0) \in H$ by the definition of o' , it suffices to show that there exists $\tilde{r}_1 > 0$ such that $H \subset A'_{\tilde{r}_1}$. Indeed, the claim will be shown with $r''_1 = \tilde{r}_1 \wedge r_0$ once we have proved it. Suppose that $H \subset A'_r$ does not hold for any $r \in (0, 1)$. Then there are sequences $(t_j, x_j, y_j) \in H$, $(t'_j, x'_j, y'_j) \in \text{Cut}_{\text{ST}}$, $j \in \mathbb{N}$ such that $|t_j - t'_j| + d_{g(t_j)}(x_j, x'_j) + d_{g(t_j)}(y_j, y'_j) \rightarrow 0$ as $j \rightarrow \infty$. We may assume that $((t_j, x_j, y_j))_j$ converges. Since $(t_j, x_j, y_j) \in H$, $x'_j, y'_j \in M_0$ holds for sufficiently large j . Thus we can take a convergent subsequence of $((t'_j, x'_j, y'_j))_j$. Since Cut_{ST} and H are closed and $d_{g(\cdot)}(\cdot, \cdot)$ is continuous, it contradicts with $H \cap \text{Cut}_{\text{ST}} = \emptyset$.

To complete the proof, we show that there exists $\delta_1 \in (0, \delta_3]$ such that $(t, o', p) \in A'_{r''_1/2}$ when $\delta \in (0, \delta_1)$. Suppose that there exists $(t', x', y') \in \text{Cut}_{\text{ST}}$ such that $|t - t'| + d_{g(t)}(o', x') + d_{g(t)}(p, y') < r''_1/2$. For any $q \in \gamma_{py'}^{(t)}$, the triangle inequality and the assertion (i) yield

$$d_{g(t)}(o, q) \leq d_{g(t)}(o, p) + d_{g(t)}(p, y') \leq e^{\kappa\delta} (R - 1 + \delta) + r''_1/2. \quad (3.6)$$

A similar observation implies $d_{g(t)}(o, q') \leq (e^{\kappa\delta} r_0 + r''_1)/2$ for $q' \in \gamma_{o'x'}^{(t)}$. Thus there is $\delta_4 = \delta_4(\kappa, R) \in (0, \delta_3]$ such that the right hand side of (3.6) is less than R and $(e^{\kappa\delta} r_0 + r''_1)/2 \leq R$ whenever $\delta \in (0, \delta_4)$. In such a case, $\gamma_{py'}^{(t)} \subset M_0$ and $\gamma_{o'x'}^{(t)} \subset M_0$ hold. Since $(t_0, o', p_0) \in A'_{r''_1}$, Lemma 2.2 yields

$$\begin{aligned} |t - t'| + d_{g(t)}(o', x') + d_{g(t)}(p, y') \\ \geq |t_0 - t'| - \delta + e^{-\kappa\delta} d_{g(t_0)}(o', x') + e^{-\kappa\delta} d_{g(t_0)}(p, y') \\ \geq e^{-\kappa\delta} r''_1 + (1 - e^{-\kappa\delta}) |t_0 - t'| - \delta - e^{-\kappa\delta} \delta \end{aligned} \quad (3.7)$$

Take $\delta_1 = \delta_1(\kappa, r''_1) \in (0, \delta_4]$ so that the right hand side of (3.7) is greater than $r''_1/2$ when $\delta \in (0, \delta_1)$. Then (3.7) is absurd for any $\delta \in (0, \delta_1)$. Thus it implies the conclusion. \square

We prepare some notations for the second variational formula for the arclength. Let $\nabla^{(t)}$ be the $g(t)$ -Levi-Civita connection and $\mathcal{R}^{(t)}$ the $g(t)$ -curvature tensor associated with $\nabla^{(t)}$. For a smooth curve γ and smooth vector fields U, V along γ , the index form $I_\gamma^{(t)}(U, V)$ is given by

$$I_\gamma^{(t)}(U, V) := \int_\gamma \left(\langle \nabla_{\dot{\gamma}}^{(t)} U, \nabla_{\dot{\gamma}}^{(t)} V \rangle_{g(t)} - \langle \mathcal{R}^{(t)}(U, \dot{\gamma}) \dot{\gamma}, V \rangle_{g(t)} \right) ds.$$

We write $I_\gamma^{(t)}(U, U) =: I_\gamma^{(t)}(U)$ for simplicity of notations. Let $G_{t,x,y}(u)$ be the solution to the following initial value problem on $[0, d(x, y)]$:

$$\begin{cases} G''_{t,x,y}(u) = -\frac{\text{Ric}_{g(t)}(\dot{\gamma}_{xy}^{(t)}(u), \dot{\gamma}_{xy}^{(t)}(u))}{m-1} G_{t,x,y}(u), \\ G_{t,x,y}(0) = 0, \ G'_{t,x,y}(0) = 1. \end{cases}$$

Note that $G_{t,x,y}(u) > 0$ for $u \in (0, d(x, y)]$ if $y \notin \text{Cut}_{g(t)}(x)$ (see [18, Proof of Lemma 9]). For simplicity, we write $G_n := G_{t_n^{(\alpha)}, o_n, X^\alpha(t_n^{(\alpha)})}$. When $X^\alpha(t_n^{(\alpha)}) \notin B_{r_0}^{(t_n^{(\alpha)})}(o)$, we define a vector field V^\dagger along γ_n for each $V \in T_{X^\alpha(t_n^{(\alpha)})}M$ by

$$V^\dagger(\gamma_n(u)) := \frac{G_n(u)}{G_n(d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)})))} (\parallel_{\gamma_n}^{(t_n^{(\alpha)})} V)(\gamma_n(u)),$$

where $\parallel_{\gamma_n}^{(t_n^{(\alpha)})} V$ is the parallel vector field along γ_n of V associated with $\nabla^{(t_n^{(\alpha)})}$. Take $v \in \mathbb{R}^m$. By using these notations, for $n \in \mathbb{N}_0$ with $n < N^{(\alpha)}$, let us define λ_{n+1} and Λ_{n+1} by

$$\begin{aligned} \lambda_{n+1} &:= \langle \tilde{\xi}_{n+1}, \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}, \\ \Lambda_{n+1} &:= \partial_t d_{g(t_n^{(\alpha)})}(o, o_n) + \partial_t d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)})) \\ &\quad + \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) + \frac{1}{2} I_{\gamma_n}^{(t_n^{(\alpha)})}(\tilde{\xi}_{n+1}^\dagger) \end{aligned}$$

when $X^\alpha(t_n^{(\alpha)}) \notin B_{r_0}^{(t_n^{(\alpha)})}(o)$, and $\lambda_{n+1} = \sqrt{m+2} \langle \xi_{n+1}, v \rangle_{\mathbb{R}^m}$ and $\Lambda_{n+1} = 0$ otherwise.

Lemma 3.7 *If $n < \bar{\sigma}_R \wedge N^{(\alpha)}$, $\alpha < \alpha_0$ is sufficiently small and $X^\alpha(t_n^{(\alpha)}) \notin B_{r_0}^{(t_n^{(\alpha)})}(o)$, then*

$$d_{g(t_{n+1}^{(\alpha)})}(o, X^\alpha(t_{n+1}^{(\alpha)})) \leq d_{g(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)})) + \alpha \lambda_{n+1} + \alpha^2 \Lambda_{n+1} + o(\alpha^2)$$

almost surely, where α_0 is as in Corollary 3.6. In addition, $o(\alpha^2)$ is controlled uniformly.

Proof. By virtue of Corollary 3.6, for sufficiently small α , the Taylor expansion together with the second variational formula yields

$$\begin{aligned} d_{g(t_{n+1}^{(\alpha)})}(o_n, X^\alpha(t_{n+1}^{(\alpha)})) &\leq d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)})) + \alpha \lambda_{n+1} + \alpha^2 \partial_t d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)})) \\ &\quad + \alpha^2 \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) + \frac{\alpha^2}{2} I_{\gamma_n}^{(t_n^{(\alpha)})}(J_{\tilde{\xi}_{n+1}}) \\ &\quad + o(\alpha^2), \end{aligned} \tag{3.8}$$

where $J_{\tilde{\xi}_{n+1}}$ is a $g(t_n^{(\alpha)})$ -Jacobi field along γ_n with the boundary value condition $J_{\tilde{\xi}_{n+1}}(o_n) = 0$ and $J_{\tilde{\xi}_{n+1}}(X^\alpha(t_n^{(\alpha)})) = \tilde{\xi}_{n+1}$. Note that $o(\alpha^2)$ can be chosen uniformly since this expansion can be done on the compact set A_{r_1} and every geodesic variation is included in M_0 . By the index lemma, we have $I_{\gamma_n}^{(t_n^{(\alpha)})}(J_{\tilde{\xi}_{n+1}}) \leq I_{\gamma_n}^{(t_n^{(\alpha)})}(\tilde{\xi}_{n+1}^\dagger)$. Hence the desired inequality follows when $o_n = o$. In the case $o_n \neq o$, we have

$$\begin{aligned} d_{g(t_{n+1}^{(\alpha)})}(o, X^\alpha(t_{n+1}^{(\alpha)})) &\leq d_{g(t_{n+1}^{(\alpha)})}(o, o_n) + d_{g(t_{n+1}^{(\alpha)})}(o_n, X^\alpha(t_{n+1}^{(\alpha)})), \\ d_{g(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)})) &= d_{g(t_n^{(\alpha)})}(o, o_n) + d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)})). \end{aligned}$$

Note that $(t_n^{(\alpha)}, o, o_n)$ is uniformly away from Cut_{ST} because of our choice of r_0 and Lemma 2.6. Therefore the conclusion follows by combining them with (3.8). \square

Before turning into the next step, we show the following two complementary lemmas (Lemma 3.8 and Lemma 3.9) which provide a nice control of the second order term Λ_n in Lemma 3.7. Set $\bar{\Lambda}_n = \mathbb{E}[\Lambda_n \mid \mathcal{F}_{n-1}]$.

Lemma 3.8 *Let $(a_n)_{n \in \mathbb{N}_0}$ be a uniformly bounded \mathcal{F}_n -predictable process. Then*

$$\lim_{\alpha \rightarrow 0} \alpha^2 \sup \left\{ \left| \sum_{j=n}^{N+1} a_j (\Lambda_j - \bar{\Lambda}_j) \right| \mid n, N \in \mathbb{N}, n \leq N \leq N^{(\alpha)} \wedge \bar{\sigma}_R \right\} = 0 \quad \text{in probability.}$$

Proof. Note that the map $(t, x, y) \mapsto G_{t,x,y}(d(x, y))$ is continuous on A_{r_1} . Since we have $G_{t,x,y}(d(x, y)) > 0$ on A_{r_1} , there exists $K > 0$ such that $K^{-1} < G_{t,x,y}(d(x, y)) < K$. This fact together with Corollary 3.6 yields $|\Lambda_j|$ and $|\bar{\Lambda}_j|$ are uniformly bounded if $j < \bar{\sigma}_R$. Since $\sum_{j=1}^n a_j (\Lambda_j - \bar{\Lambda}_j)$ is an \mathcal{F}_n -martingale and $\bar{\sigma}_R$ is \mathcal{F}_n -stopping time, the Doob inequality yields

$$\lim_{\alpha \rightarrow 0} \alpha^2 \sup_{0 \leq N \leq N^{(\alpha)} \wedge \bar{\sigma}_R} \left| \sum_{j=1}^{N+1} a_j (\Lambda_j - \bar{\Lambda}_j) \right| = 0 \quad \text{in probability.} \quad (3.9)$$

Here we used the fact $\lim_{\alpha \rightarrow 0} \alpha^2 N^{(\alpha)} = T_2 - T_1$. Note that

$$\begin{aligned} & \bigcup_{N=1}^{N^{(\alpha)} \wedge \bar{\sigma}_R} \bigcup_{n=1}^N \left\{ \alpha^2 \left| \sum_{j=n}^{N+1} a_j (\Lambda_j - \bar{\Lambda}_j) \right| > \delta \right\} \\ & \subset \bigcup_{N=1}^{N^{(\alpha)} \wedge \bar{\sigma}_R} \bigcup_{n=2}^N \left\{ \alpha^2 \left| \sum_{j=1}^{n-1} a_j (\Lambda_j - \bar{\Lambda}_j) \right| > \frac{\delta}{2} \right\} \cup \left\{ \alpha^2 \left| \sum_{j=1}^{N+1} a_j (\Lambda_j - \bar{\Lambda}_j) \right| > \frac{\delta}{2} \right\} \\ & = \left\{ \alpha^2 \sup_{0 \leq N \leq N^{(\alpha)} \wedge \bar{\sigma}_R} \left| \sum_{j=0}^N a_j (\Lambda_j - \bar{\Lambda}_j) \right| > \frac{\delta}{2} \right\}. \end{aligned}$$

Thus the conclusion follows from (3.9). \square

Lemma 3.9 *There exists a deterministic constant $C_0 > 0$ being independent of α and R such that the following holds:*

$$\bar{\Lambda}_{n+1} \leq C_0 + \frac{1}{2} \int_0^d g^{(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)})) b(u) du.$$

Proof. By using $(m+2)\mathbb{E}[\langle \xi_n, e_i \rangle \langle \xi_n, e_j \rangle] = \delta_{ij}$, we obtain

$$\begin{aligned} \mathbb{E} \left[I_{\gamma_n}^{(t_n^{(\alpha)})}(\tilde{\xi}_{n+1}^\dagger) \right] &= \sum_{j=2}^m I_{\gamma_n}^{(t_n^{(\alpha)})} \left(\left(\Phi^{(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) e_j \right)^\dagger \right) \\ &= \frac{(m-1)G'_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}{G_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}. \end{aligned}$$

Note that we have

$$\begin{aligned}
& \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) - \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(o_n) \\
&= \int_0^{d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)}))} \partial_s \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(\gamma_n(s))|_{s=u} du \\
&= \int_0^{d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)}))} \langle \nabla_{\dot{\gamma}_n}^{(t_n^{(\alpha)})} Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(\gamma_n(u)) du.
\end{aligned}$$

Recall that, for $(t, x, y) \notin \text{Cut}_{\text{ST}}$, we have

$$\partial_t d_{g(t)}(x, y) = \frac{1}{2} \int_0^{d_{g(t)}(x, y)} (\partial_t g(t))(\dot{\gamma}_{xy}^{(t)}(u), \dot{\gamma}_{xy}^{(t)}(u)) du$$

(cf. [19, Remark 6]). By combining them with Assumption 1,

$$\begin{aligned}
\bar{\Lambda}_{n+1} &= \partial_t d_{g(t_n^{(\alpha)})}(o, o_n) + \frac{1}{2} \int_0^{d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)}))} \partial_t g(t_n^{(\alpha)})(\dot{\gamma}_n(u), \dot{\gamma}_n(u)) du \\
&\quad + \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(X^\alpha(t_n^{(\alpha)})) + \frac{(m-1)G'_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}{2G_n(d(o_n, X^\alpha(t_n^{(\alpha)})))} \\
&\leq \frac{1}{2} \int_0^{d_{g(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)}))} b(u) du + \partial_t d_{g(t_n^{(\alpha)})}(o, o_n) + \langle Z(t_n^{(\alpha)}), \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})}(o_n) \\
&\quad + \frac{1}{2} \int_0^{d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)}))} \text{Ric}_{g(t_n^{(\alpha)})}(\dot{\gamma}_n(u), \dot{\gamma}_n(u)) du \\
&\quad + \frac{(m-1)G'_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}{2G_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}. \tag{3.10}
\end{aligned}$$

Here we used the fact $b(u) \geq 0$ in the case $o_n \neq o$. Note that

$$\int_0^r \text{Ric}_{g(t_n^{(\alpha)})}(\dot{\gamma}_n(u), \dot{\gamma}_n(u)) du + \frac{(m-1)G'_n(r)}{G_n(r)}$$

is non-increasing as a function of r . Indeed, we can easily verify it by taking a differentiation. Set

$$C_1 := \sup_{t \in [T_1, T_2]} \sup_{x \in B_{r_0}^{(t)}(o)} \left(|Z(t)|_{g(t)}(x) + \sup_{\substack{V \in T_x M \\ |V|_{g(t)} \leq 1}} (\partial_t g(t)(V, V) + |\text{Ric}_{g(t)}(V, V)|) \right).$$

By virtue of Lemma 2.2, $C_1 < \infty$ holds. By applying a usual comparison argument to $G'_n(r_0)/G_n(r_0)$, we obtain

$$\begin{aligned}
& \int_0^{d_{g(t_n^{(\alpha)})}(o_n, X^\alpha(t_n^{(\alpha)}))} \text{Ric}_{g(t_n^{(\alpha)})}(\dot{\gamma}_n(u), \dot{\gamma}_n(u)) du + \frac{(m-1)G'_n(d(o_n, X^\alpha(t_n^{(\alpha)})))}{G_n(d(o_n, X^\alpha(t_n^{(\alpha)})))} \\
&\leq C_1 (r_0 + \coth(C_1 r_0)).
\end{aligned}$$

Hence the conclusion follows from (3.10) with $C_0 = C_1(1 + 3r_0/4 + \coth(C_1 r_0)/2)$. \square

In the next step, we will introduce a comparison process to give a control of the radial process. Let us define two functions φ and ψ on $(2r_0, \infty)$ by

$$\begin{aligned}\varphi(r) &:= C_0 + \frac{1}{2} \int_0^r b(u) du, \\ \psi(r) &:= \frac{2}{r - 2r_0},\end{aligned}$$

where C_0 is as in Lemma 3.9. Let us define a comparison process $\rho^\alpha(t)$ taking values in $[0, \infty)$ inductively by

$$\begin{aligned}\rho^\alpha(0) &:= d_{g(0)}(o, x_0) + 3r_0, \\ \rho^\alpha(t) &:= \rho^\alpha(t_n^{(\alpha)}) + \frac{t - t_n^{(\alpha)}}{\alpha^2} \left(\alpha \lambda_{n+1} + \alpha^2 (\varphi(\rho^\alpha(t_n^{(\alpha)})) + \psi(\rho^\alpha(t_n^{(\alpha)}))) \right), \quad t \in [t_n^{(\alpha)}, t_{n+1}^{(\alpha)}].\end{aligned}$$

The term $\psi(\rho^\alpha(t_n^{(\alpha)}))$ is inserted to avoid a difficulty coming from the absence of the estimate in Lemma 3.7 on a neighborhood of o . By virtue of this extra term, $\rho^\alpha(t) > 2r_0$ holds for all $t \in [T_1, T_2]$ if α is sufficiently small. Let $\hat{\sigma}'_R$ and $\bar{\sigma}'_R$ be given by $\hat{\sigma}'_R := \sigma_R(\rho^\alpha)$ and $\bar{\sigma}'_R := \alpha^{-2}(\lfloor \hat{\sigma}'_R \rfloor_\alpha - T_1) + 1$. The following is a modification of an argument in the proof of [11, Theorem 3.5.3] into our discrete setting.

Lemma 3.10 *For $\delta > 0$, there exist a family of events $(E_\delta^\alpha)_\alpha$ with $\lim_{\alpha \rightarrow 0} \mathbb{P}[E_\delta^\alpha] = 1$ and a constant $K(\delta) > 0$ with $\lim_{\delta \rightarrow 0} K(\delta) = 0$ such that, on E_δ^α ,*

$$d_{g(t)}(o, X^\alpha(t)) \leq \rho^\alpha(t) + K(\delta)$$

for $t \in [T_1, \hat{\sigma}_R \wedge \hat{\sigma}'_R \wedge T_2]$ and sufficiently small α relative to δ and R^{-1} .

Proof. It suffices to show the assertion in the case $t = t_n^{(\alpha)}$ for some $n \in \mathbb{N}_0$. Indeed, once we have shown it, Corollary 3.6 (i) yields

$$\begin{aligned}d_{g(t)}(o, X^\alpha(t)) &\leq e^{\kappa\alpha^2} (d_{g(\lfloor t \rfloor_\alpha)}(o, X^\alpha(\lfloor t \rfloor_\alpha)) + h(\alpha)) \\ &\leq \rho_{\lfloor t \rfloor_\alpha}^\alpha + K(\delta) + (e^{\kappa\alpha^2} - 1)R + e^{\kappa\alpha^2} h(\alpha) \\ &\leq \rho_t^\alpha + K(\delta) + \alpha + (e^{\kappa\alpha^2} - 1)R + e^{\kappa\alpha^2} h(\alpha)\end{aligned}$$

for $t \in [T_1, \hat{\sigma}_R \wedge T_2]$. Here we used the fact $\varphi \geq 0$ and $\psi > 0$. Thus the conclusion can be easily deduced.

For simplicity of notations, we denote $d_{g(t_n^{(\alpha)})}(o, X^\alpha(t_n^{(\alpha)}))$ and $\rho^\alpha(t_n^{(\alpha)})$ by d_n and ρ_n respectively in the rest of this proof. Let us define a sequence of \mathcal{F}_n -stopping times S_l by $S_0 := 0$ and

$$\begin{aligned}S_{2l+1} &:= \inf \left\{ j \geq S_{2l} \mid X^\alpha(t_j^{(\alpha)}) \in B_{r_0}^{(t_j^{(\alpha)})}(o) \right\} \wedge N^{(\alpha)}, \\ S_{2l} &:= \inf \left\{ j \geq S_{2l-1} \mid X^\alpha(t_j^{(\alpha)}) \notin B_{3r_0/2}^{(t_j^{(\alpha)})}(o) \right\} \wedge N^{(\alpha)}.\end{aligned}$$

Since $\rho_n > 2r_0$, it suffices to show the assertion in the case $S_{2l} \leq n < S_{2l+1} \wedge \bar{\sigma}_R \wedge \bar{\sigma}'_R$ for some $l \in \mathbb{N}_0$. Now Lemma 3.7 and Lemma 3.9 imply

$$d_{j+1} - \rho_{j+1} \leq d_j - \rho_j + \alpha^2(\varphi(d_j) - \varphi(\rho_j)) + \alpha^2(\Lambda_{j+1} - \bar{\Lambda}_{j+1}) + o(\alpha^2)$$

for $j \in [S_{2l}, S_{2l+1} \wedge \sigma'_R \wedge \bar{\sigma}'_R)$. Here we used the fact $\psi > 0$. Let f_α be a C^2 -function on \mathbb{R} satisfying

- (i) $f_\alpha|_{(-\infty, -\alpha)} \equiv 0$, $f_\alpha|_{(\alpha, \infty)}(x) = x$,
- (ii) f_α is convex,
- (iii) $\alpha^2 \sup_{x \in \mathbb{R}} f''_\alpha(x) = O(1)$.

For example, a function f_α satisfying these conditions is constructed by setting

$$\tilde{f}(x) = \int_{-\infty}^x \int_{-\infty}^t b \exp\left(-\frac{a}{1-s^2}\right) 1_{(-1,1)}(s) ds dt,$$

where a, b is chosen to satisfy

$$\int_{-\infty}^1 \exp\left(-\frac{a}{1-s^2}\right) 1_{(-1,1)}(s) ds = 1, \quad b \int_{-\infty}^1 \int_{-\infty}^t \exp\left(-\frac{a}{1-s^2}\right) 1_{(-1,1)}(s) ds dt = 1$$

and $f_\alpha(x) := \alpha \tilde{f}(\alpha^{-1}x)$. By the Taylor expansion with the condition (iii) of f_α , we have

$$\begin{aligned} f_\alpha(d_{j+1} - \rho_{j+1}) &\leq f_\alpha(d_j - \rho_j) \\ &\quad + \alpha^2 f'_\alpha(d_j - \rho_j)(\varphi(d_j) - \varphi(\rho_j) + (\Lambda_j - \bar{\Lambda}_j)) + o(\alpha^2). \end{aligned} \quad (3.11)$$

Let $C > 0$ be the Lipschitz constant of φ on $[0, R]$. Note that we have

$$f'_\alpha(d_j - \rho_j)(\varphi(d_j) - \varphi(\rho_j)) \leq C(d_j - \rho_j)_+ + o(1). \quad (3.12)$$

Here the error term $o(1)$ may appear in the case $d_j - \rho_j \in [-\alpha, 0]$. Now by using (3.11) and (3.12) combined with the fact $d_{S_{2l}} - \rho_{S_{2l}} < -\alpha$ for sufficiently small α , we obtain

$$\begin{aligned} (d_n - \rho_n)_+ &\leq f_\alpha(d_n - \rho_n) \\ &\leq C\alpha^2 \sum_{j=S_{2k}}^{n-1} (d_j - \rho_j)_+ + \alpha^2 \sum_{j=S_{2k}}^{n-1} f'_\alpha(d_j - \rho_j)(\Lambda_{j+1} - \bar{\Lambda}_{j+1}) + o(1). \end{aligned} \quad (3.13)$$

Here the first inequality follows from the condition (ii) of f_α and $n \leq \alpha^{-2}(T_2 - T_1)$ is used to derive the error term $o(1)$. Let E_δ^α be an event defined by

$$E_\delta^\alpha := \left\{ \alpha^2 \sup_{k \leq k' \leq N^{(\alpha)} \wedge \bar{\sigma}_R} \left| \sum_{j=k}^{k'} f'_\alpha(d_{j-1} - \rho_{j-1})(\Lambda_j - \bar{\Lambda}_j) \right| < \delta \right\}.$$

Note that $a_j = f'_\alpha(d_{j-1} - \rho_{j-1})$ is \mathcal{F}_n -predictable and uniformly bounded by 1. Thus, by combining Lemma 3.8 with (3.13), we obtain

$$(d_n - \rho_n)_+ \leq C\alpha^2 \sum_{j=S_{2l}}^{n-1} (d_j - \rho_j)_+ + 2\delta$$

on E_δ^α for sufficiently small α . Thus, by virtue of a discrete Gronwall inequality (see [28] for instance),

$$(d_n - \rho_n)_+ \leq 2\delta (1 + (1 + C\alpha^2)^n) \leq 2\delta(1 + e^{C(T_2 - T_1)}).$$

This estimate implies the conclusion. \square

Corollary 3.11 *For every $R' < R$, $\limsup_{\alpha \rightarrow 0} \mathbb{P}[\hat{\sigma}_R \leq T_2] \leq \limsup_{\alpha \rightarrow 0} \mathbb{P}[\hat{\sigma}'_{R'} \leq T_2]$.*

Now we turn to the proof of our destination in this section.

Proof of Proposition 3.4. By Corollary 3.11, the proof of Proposition 3.4 is reduced to estimate $\mathbb{P}[\hat{\sigma}'_R \leq T_2]$. To obtain a useful bound of it, we would like to apply the invariance principle for ρ^α . However, there is a technical difficulty coming from the unboundedness of the drift term of ρ^α . To avoid it, we introduce an auxiliary process $\tilde{\rho}^\alpha$ in the sequel.

Let $\tilde{\varphi}$ be a bounded, globally Lipschitz function on \mathbb{R} such that $\tilde{\varphi}(r) = \varphi(r) + \psi(r)$ for $r \in [2r_0 + R^{-1}, R]$. Let us define an \mathbb{R} -valued process $\tilde{\rho}^\alpha(t)$ inductively by

$$\begin{aligned} \tilde{\rho}^\alpha(0) &:= d_{g(T_1)}(o, x_0) + 3r_0, \\ \tilde{\rho}^\alpha(t) &:= \tilde{\rho}^\alpha(t_n^{(\alpha)}) + \frac{t - t_n^{(\alpha)}}{\alpha^2} (\alpha\lambda_{n+1} + \alpha^2\tilde{\varphi}(\rho^\alpha(t_n^{(\alpha)}))), \quad t \in [t_n^{(\alpha)}, t_{n+1}^{(\alpha)}]. \end{aligned}$$

We also define two diffusion processes $\rho^0(t)$ and $\tilde{\rho}^0(r)$ as solutions to the following SDEs:

$$\begin{cases} d\rho^0(t) = dB(t) + (\varphi(\rho^0(t)) + \psi(\rho^0(t)))dt, \\ \rho^0(T_1) = d_{g(T_1)}(o, x_0) + 3r_0, \\ d\tilde{\rho}^0(t) = dB(t) + \tilde{\varphi}(\tilde{\rho}^0(t))dt, \\ \tilde{\rho}^0(T_1) = d_{g(T_1)}(o, x_0) + 3r_0, \end{cases}$$

where $(B(t))_{t \in [T_1, T_2]}$ is a standard 1-dimensional Brownian motion with $B(T_1) = 0$. We claim that $\tilde{\rho}^\alpha$ converges in law to $\tilde{\rho}^0$ as $\alpha \rightarrow 0$. Indeed, we can easily show the tightness of $(\rho^\alpha)_{\alpha > 0}$ by modifying an argument for the invariance principle for i.i.d. sequences since $\tilde{\varphi}$ is bounded. Then the claim follows from the same argument as we used in the proof of Theorem 3.1 under Proposition 3.2, which is based on the Poisson subordination and the uniqueness of the martingale problem.

Let us define $\eta_R : \mathcal{C}_1 \rightarrow [T_1, T_2] \cup \{\infty\}$ by $\eta_R(w) = \inf \{t \in [T_1, T_2] \mid w(t) \leq 2r_0 + R^{-1}\}$. Then we have

$$\mathbb{P}[\hat{\sigma}'_R \leq T_2] \leq \mathbb{P}[\sigma_R(\rho^\alpha) \wedge \eta_R(\rho^\alpha) \leq T_2] = \mathbb{P}[\sigma_R(\tilde{\rho}^\alpha) \wedge \eta_R(\tilde{\rho}^\alpha) \leq T_2].$$

Since $\{w \mid \sigma_R(w) \wedge \eta_R(w) \leq T_2\}$ is closed in \mathcal{C}_1 , the Portmanteau theorem implies

$$\limsup_{\alpha \rightarrow 0} \mathbb{P}[\sigma_R(\tilde{\rho}^\alpha) \wedge \eta_R(\tilde{\rho}^\alpha) \leq T_2] \leq \mathbb{P}[\sigma_R(\tilde{\rho}^0) \wedge \eta_R(\tilde{\rho}^0) \leq T_2] = \mathbb{P}[\sigma_R(\rho^0) \wedge \eta_R(\rho^0) \leq T_2].$$

Since ρ^0 is a diffusion process on $(2r_0, \infty)$ which cannot reach the boundary by Assumption 1, the conclusion follows. \square

3.2 TIGHTNESS OF GEODESIC RANDOM WALKS

Recall that we have metrized the path space \mathcal{C} by using $d_{g(T_1)}$. To deal with the tightness of $(X^\alpha)_{\alpha \in (0,1)}$ in \mathcal{C} , we show the following lemma, which provides a tightness criterion compatible with the time-dependent metric $d_{g(t)}$.

Lemma 3.12 $(X^\alpha)_{\alpha \in (0,1)}$ is tight if

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \limsup_{\alpha \rightarrow \infty} \sup_{n \in \mathbb{N}_0} \mathbb{P} \left[\sup_{t_n^{(\alpha)} \leq s \leq (t_n^{(\alpha)} + \delta) \wedge T_2} d_{g(s)}(X^\alpha(t_n^{(\alpha)}), X^\alpha(s)) > \varepsilon, \hat{\sigma}_R = \infty \right] = 0 \quad (3.14)$$

holds for every $\varepsilon > 0$ and $R > 1$.

Proof. By following a standard argument (cf. [4, Theorem 7.3 and Theorem 7.4]), we can easily show that $(X^\alpha)_{\alpha \in (0,1)}$ is tight if, for every $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \limsup_{\alpha \rightarrow 0} \sup_{t \in [T_1, T_2]} \mathbb{P} \left[\sup_{t \leq s \leq (t+\delta) \wedge T_2} d_{g(T_1)}(X^\alpha(t), X^\alpha(s)) > \varepsilon \right] = 0.$$

Thus, by virtue of Proposition 3.4, $(X^\alpha)_{\alpha \in (0,1)}$ is tight if

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \limsup_{\alpha \rightarrow 0} \sup_{t \in [T_1, T_2]} \mathbb{P} \left[\sup_{t \leq s \leq (t+\delta) \wedge T_2} d_{g(T_1)}(X^\alpha(t), X^\alpha(s)) > \varepsilon, \hat{\sigma}_R = \infty \right] = 0$$

for every $\varepsilon > 0$ and $R > 1$. Given $R > 1$, take M_0 and κ as in Lemma 2.4 and Lemma 2.2 respectively. Then, for $\varepsilon < 1$ and $s, t \in [T_1, T_2]$,

$$\{d_{g(s)}(X^\alpha(s), X^\alpha(\lfloor t \rfloor_\alpha)) \leq \varepsilon, \hat{\sigma}_R = \infty\} \subset \{d_{g(T_1)}(X^\alpha(s), X^\alpha(t)) \leq 2e^{\kappa(T_2-T_1)}\varepsilon, \hat{\sigma}_R = \infty\}$$

if α is sufficiently small. Thus we have

$$\begin{aligned} & \left\{ \sup_{t \leq s \leq (t+\delta) \wedge T_2} d_{g(T_1)}(X^\alpha(t), X^\alpha(s)) > \varepsilon, \hat{\sigma}_R = \infty \right\} \\ & \subset \left\{ \sup_{\lfloor t \rfloor_\alpha \leq s \leq (\lfloor t \rfloor_\alpha + 2\delta) \wedge T_2} d_{g(s)}(X^\alpha(\lfloor t \rfloor_\alpha), X^\alpha(s)) > \frac{e^{-\kappa(T_2-T_1)}\varepsilon}{2}, \hat{\sigma}_R = \infty \right\} \end{aligned}$$

for $\alpha^2 \leq \delta$ and hence the conclusion follows. \square

Proof of Proposition 3.2. Take $R > 1$. By virtue of Lemma 3.12, it suffices to show (3.14). Take $M_0 \subset M$ compact and κ as in Lemma 2.4 and Lemma 2.2 respectively. By taking smaller $\varepsilon > 0$, we may assume that $\varepsilon < \tilde{r}_0/2$, where $\tilde{r}_0 = \tilde{r}_0(M_0)$ is as in Lemma 2.6. Take $n \in \mathbb{N}_0$ with $n < N^{(\alpha)}$. Let us define a \mathcal{F}_k -stopping time ζ_ε by

$$\zeta_\varepsilon := \inf \left\{ k \in \mathbb{N}_0 \mid n \leq k \leq N^{(\alpha)}, d_{g(t_k^{(\alpha)})}(X^\alpha(t_n^{(\alpha)}), X^\alpha(t_k^{(\alpha)})) > \varepsilon \right\}.$$

Then, for sufficiently small α ,

$$\left\{ \sup_{t_n^{(\alpha)} \leq s \leq (t_n^{(\alpha)} + \delta) \wedge T_2} d_{g(s)}(X^\alpha(t_n^{(\alpha)}), X^\alpha(s)) \geq 2\varepsilon, \hat{\sigma}_R = \infty \right\} \subset \{ \alpha^2(\zeta_\varepsilon - n) < \delta, \hat{\sigma}_R = \infty \}. \quad (3.15)$$

Set $p_k := X^\alpha(t_k^{(\alpha)})$ for $k \in \mathbb{N}_0$ and $f(t, x) := d_{g(t)}(p_n, x)$. Note that f^2 is smooth on $\{f < \varepsilon\}$. Let us define λ'_k by

$$\lambda'_{k+1} := \langle \tilde{\xi}_{k+1}, \dot{\gamma}_{p_n p_k}^{(t_k^{(\alpha)})} \rangle_{g(t_k^{(\alpha)})}.$$

We claim that there exists a constant $C > 0$ such that

$$f(t_{k+1}^{(\alpha)}, p_{k+1})^2 \leq f(t_k^{(\alpha)}, p_k)^2 + 2\alpha f(t_k^{(\alpha)}, p_k) \lambda'_{k+1} + C\alpha^2 \quad (3.16)$$

for $k \leq \zeta_\varepsilon \wedge N^{(\alpha)}$ on $\{\hat{\sigma}_R = \infty\}$. Indeed, in the same way as we did to obtain (3.8),

$$\begin{aligned} f(t_{k+1}^{(\alpha)}, p_{k+1})^2 &\leq f(t_k^{(\alpha)}, p_k)^2 + 2\alpha f(t_k^{(\alpha)}, p_k) \lambda'_{k+1} + \alpha^2 (\lambda'_{k+1})^2 \\ &\quad + 2\alpha^2 f(t_k^{(\alpha)}, p_k) \left(\partial_t f(t_k^{(\alpha)}, p_k) + \langle Z(t_k^{(\alpha)}), \dot{\gamma}_{p_n p_k}^{(t_k^{(\alpha)})} \rangle_{g(t_k^{(\alpha)})}(p_k) \right) \\ &\quad + \alpha^2 f(t_k^{(\alpha)}, p_k) I_{\gamma_{p_n p_k}^{(t_k^{(\alpha)})}}^{(t_k^{(\alpha)})}(J_{\tilde{\xi}_{k+1}}) + o(\alpha^2). \end{aligned} \quad (3.17)$$

Here $o(\alpha^2)$ is controlled uniformly. Let $K_1 > 0$ be a constant satisfying that the $g(t)$ -sectional curvature on M_0 is bounded below by $-K_1$ for every $t \in [T_1, T_2]$. Such a constant exists since M_0 is compact. Then a comparison argument implies

$$f(t_k^{(\alpha)}, p_k) I_{\gamma_{p_n p_k}^{(t_k^{(\alpha)})}}^{(t_k^{(\alpha)})}(J_{\tilde{\xi}_{k+1}}) \leq K_1 f(t_k^{(\alpha)}, p_k) \coth(K_1 f(t_k^{(\alpha)}, p_k)).$$

Here the right hand side is bounded uniformly if $k < \zeta_\varepsilon \wedge N^{(\alpha)}$. The remaining estimate of the second order term in (3.17) to show (3.16) is easy since we are on the event $\{\hat{\sigma}_R = \infty\}$. Applying (3.16) repeatedly from $k = n$ to $k = \zeta_\varepsilon$, we obtain

$$\varepsilon^2 < \alpha \sum_{k=n}^{\zeta_\varepsilon} f(t_k^{(\alpha)}, p_k) \lambda'_{k+1} + C\delta$$

on $\{\alpha^2(\zeta_\varepsilon - n) < \delta, \hat{\sigma}_R = \infty\}$. Set $N_\delta^{(\alpha)} := \sup\{k \in \mathbb{N}_0 \mid k \leq \alpha^{-2}\delta + n\}$. By taking $\delta < (2C)^{-1}\varepsilon^2$, we obtain

$$\begin{aligned} \{ \alpha^2(\zeta_\varepsilon - n) < \delta, \hat{\sigma}_R = \infty \} &\subset \left\{ \sum_{k=n}^{\zeta_\varepsilon} f(t_k^{(\alpha)}, p_k) \lambda'_{k+1} > \frac{\varepsilon^2}{2\alpha}, \alpha^2(\zeta_\varepsilon - n) < \delta, \hat{\sigma}_R = \infty \right\} \\ &\subset \left\{ \sup_{n \leq N \leq N_\delta^{(\alpha)}} \sum_{k=n}^N f(t_k^{(\alpha)}, p_k) 1_{\{f(t_k^{(\alpha)}, p_k) \leq \varepsilon\}} \lambda'_{k+1} > \frac{\varepsilon^2}{2\alpha} \right\}. \end{aligned} \quad (3.18)$$

Set

$$Y_{k+1} := \frac{1}{\sqrt{m+2}} f(t_k^{(\alpha)}, p_k) 1_{\{f(t_k^{(\alpha)}, p_k) \leq \varepsilon\}} \lambda'_{k+1}.$$

We can easily see that $|Y_k| \leq 1$ and $\sum_{k=n+1}^N Y_k$ is \mathcal{F}_N -martingale. By [10, Theorem 1.6] with (3.18), we obtain

$$\begin{aligned} \mathbb{P}[\alpha^2(\zeta_\varepsilon - n) < \delta, \hat{\sigma}_R = \infty] &\leq \mathbb{P}\left[\sup_{n \leq N \leq N_\delta^{(\alpha)}} \sum_{k=n+1}^{N+1} Y_k > \frac{\varepsilon^2}{2\alpha\sqrt{m+2}}\right] \\ &\leq \exp\left(-\frac{\varepsilon^4}{4\sqrt{m+2}(\alpha\varepsilon^2 + 2\alpha^2\sqrt{m+2}(N_\delta^{(\alpha)} - n))}\right) \\ &\leq \exp\left(-\frac{\varepsilon^4}{4\sqrt{m+2}(\alpha\varepsilon^2 + 2\sqrt{m+2}\delta)}\right). \end{aligned}$$

Hence (3.14) follows by combining this estimate with (3.15). \square

4 COUPLING BY REFLECTION

For $k \in \mathbb{R}$, let $U_{a,k}$ be a 1-dimensional Ornstein-Uhlenbeck process defined as a solution to the following SDE:

$$\begin{aligned} dU_{a,k}(t) &= -\frac{k}{2}U_{a,k}(t)dt + 2dB(t), \\ U_{a,k}(T_1) &= a. \end{aligned}$$

More explicitly, $U_{a,k}(t) = e^{-k(t-T_1)/2}a + 2\int_0^t e^{k(s-t)/2}dB(s)$. Here $B(t)$ is standard 1-dimensional Brownian motion as in the proof of Proposition 3.4.

Theorem 4.1 *Suppose*

$$2(\nabla Z(t))^\flat + \partial_t g(t) \leq \text{Ric}_{g(t)} + kg(t) \quad (4.1)$$

holds for some $k \in \mathbb{R}$. Then, for each $x_1, x_2 \in M$, there exists a coupling $\mathbf{X}(t) := (X_1(t), X_2(t))$ of two \mathcal{L}_t -diffusion particles starting at (x_1, x_2) satisfying

$$\mathbb{P}\left[\inf_{T_1 \leq t \leq T} d_{g(t)}(\mathbf{X}(t)) > 0\right] \leq \mathbb{P}\left[\inf_{T_1 \leq t \leq T} U_{d_{g(T_1)}(x_1, x_2), k}(t) > 0\right] = \chi\left(\frac{d_{g(T_1)}(x_1, x_2)}{2\sqrt{\beta(T-T_1)}}\right)$$

for each $T \in [T_1, T_2]$, where

$$\chi(a) := \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{-u^2/2} du, \quad \beta(t) := \begin{cases} \frac{e^{kt} - 1}{k} & k \neq 0, \\ t & k = 0. \end{cases}$$

- Remark 4.2** (i) Given $k \in \mathbb{R}$, a simple example satisfying (4.1) can be constructed from a solution $\tilde{g}(t)$ to the Ricci flow $\partial_t \tilde{g}(t) = \text{Ric}_{\tilde{g}(t)}$ by a scaling. That is, $g(t) = e^{-kt} \tilde{g}(t)$ satisfies (4.1) (with equality) when $Z(t) \equiv 0$.
- (ii) Our assumption (4.1) can be regarded as a natural extension of a lower Ricci curvature bound by k . Indeed, Bakry-Émery's curvature-dimension condition $\text{CD}(k, \infty)$ (see [2] for instance), which is a natural extension of a lower Ricci curvature bound, appears in (4.1) when both $Z(t)$ and $g(t)$ are independent of t .
- (iii) From the last item in this remark, when $Z(t) \equiv 0$, one may expect that (4.1) works as an analogue of Bakry-Émery's $\text{CD}(k, N)$ condition, which is equivalent to $\text{Ric}_g \geq k$ and $\dim M < N$ when $g(t)$ is independent of t , instead of $\text{CD}(k, \infty)$ since $\dim M = m < \infty$ in our case. However, the following observation suggests us that we should be more careful: Let us consider (4.1) in the case $k > 0$ and $Z(t) \equiv 0$. When $\partial_t g(t) \equiv 0$, the Bonnet-Myers theorem tells us that the diameter of M is bounded and hence M is compact. When $g(t)$ depends on t , it is no longer true. In fact, we can easily obtain a noncompact M satisfying (4.1) with $k > 0$ by following an observation in the first item of this remark. On the other hand, the Bonnet-Myers theorem is known to hold under $\text{CD}(k, N)$ when Z is of the form ∇h in the time-homogeneous case (see [3, 22]).

By following a standard argument, Theorem 4.1 implies the following estimate for a gradient of the diffusion semigroup:

Corollary 4.3 Let $(\{X(t)\}_{t \in [T_1, T_2]}, \{\mathbb{P}_x\}_{x \in M})$ be a \mathcal{L}_t -diffusion process with $\mathbb{P}_x[X(T_1) = x] = 1$. For any bounded measurable function f on M , let us define $P_t f$ by $P_t f(x) := \mathbb{E}_x[f(X(t))]$. Then, under the same assumption as in Theorem 4.1, we have

$$\limsup_{y \rightarrow x} \left| \frac{P_t f(x) - P_t f(y)}{d_{g(T_1)}(x, y)} \right| \leq \frac{1}{\sqrt{2\pi\beta(t - T_1)}} \sup_{z, z' \in M} |f(z) - f(z')|.$$

In particular, $P_t f$ is $d_{g(T_1)}$ -globally Lipschitz continuous when f is bounded.

Proof of Corollary 4.3. Let $\mathbf{X} = (X_1, X_2)$ be a coupling of \mathcal{L}_t -diffusions $(X(t), \mathbb{P}_x)$ and $(X(t), \mathbb{P}_y)$ given in Theorem 4.1. Let τ^* be the coupling time of \mathbf{X} , i.e. $\tau^* := \inf\{t \in [T_1, T_2] \mid \mathbf{X}(t) \in D(M)\}$. Let us define a new coupling $\mathbf{X}^* = (X_1^*, X_2^*)$ by

$$\mathbf{X}^*(t) := \begin{cases} \mathbf{X}(t) & \text{if } \tau^* > t, \\ (X_1(t), X_1(t)) & \text{otherwise.} \end{cases}$$

Since $\{\tau^* > T\} = \{\inf_{T_1 \leq t \leq T} d_{g(t)}(\mathbf{X}(t)) > 0\}$, Theorem 4.1 yields

$$\begin{aligned} P_t f(x) - P_t f(y) &= \mathbb{E}[f(X_1^*(t)) - f(X_2^*(t))] \\ &= \mathbb{E}[(f(X_1^*(t)) - f(X_2^*(t))) 1_{\{\tau^* > t\}}] \\ &\leq \mathbb{P}[\tau^* > t] \sup_{z, z' \in M} |f(z) - f(z')| \\ &\leq \chi \left(\frac{d_{g(T_1)}(x, y)}{2\sqrt{\beta(t - T_1)}} \right) \sup_{z, z' \in M} |f(z) - f(z')|. \end{aligned}$$

Hence the assertion holds by dividing the both sides of the above inequality by $d_{g(T_1)}(x, y)$ and by letting $y \rightarrow x$ after that. \square

As we did in the last section, let $(\gamma_{xy}^{(t)})_{x,y \in M}$ be a measurable family of unit-speed minimal $g(t)$ -geodesics such that $\gamma_{xy}^{(t)}$ joins x and y . Without loss of generality, we may assume that $\gamma_{xy}^{(t)}$ is symmetric, that is, $\gamma_{xy}^{(t)}(d_{g(t)}(x, y) - s) = \gamma_{yx}^{(t)}(s)$ holds. Let us define $\tilde{m}_{xy}^{(t)} : T_y M \rightarrow T_y M$ by

$$\tilde{m}_{xy}^{(t)} v := v - 2\langle v, \dot{\gamma}_{xy}^{(t)} \rangle_{g(t)} \dot{\gamma}_{xy}^{(t)}(d_{g(t)}(x, y)).$$

This is a reflection with respect to a hyperplane which is $g(t)$ -perpendicular to $\dot{\gamma}_{xy}^{(t)}$. Let us define $m_{xy}^{(t)} : T_x M \rightarrow T_y M$ by $m_{xy}^{(t)} := \tilde{m}_{xy}^{(t)} \circ \parallel_{\gamma_{xy}^{(t)}}^{(t)}$. Clearly $m_{xy}^{(t)}$ is a $g(t)$ -isometry. As in the last section, let $\Phi^{(t)} : M \rightarrow \mathcal{O}^{(t)}(M)$ be a measurable section of the $g(t)$ -orthonormal frame bundle $\mathcal{O}^{(t)}(M)$ of M . Let us define two measurable maps $\Phi_i^{(t)} : M \times M \rightarrow \mathcal{O}^{(t)}(M)$ for $i = 1, 2$ by

$$\begin{aligned} \Phi_1^{(t)}(x, y) &:= \Phi^{(t)}(x), \\ \Phi_2^{(t)}(x, y) &:= \begin{cases} m_{xy}^{(t)} \Phi_1^{(t)}(x, y), & (x, y) \in M \times M \setminus D(M), \\ \Phi^{(t)}(x), & (x, y) \in D(M). \end{cases} \end{aligned}$$

Take $x_1, x_2 \in M$. By using $\Phi_i^{(t)}$, we define a coupled geodesic random walk $\mathbf{X}^\alpha(t) = (X_1^\alpha(t), X_2^\alpha(t))$ by $X_i^\alpha(0) = x_i$ and, for $t \in [t_n^{(\alpha)}, t_{n+1}^{(\alpha)}]$,

$$\begin{aligned} \tilde{\xi}_{n+1}^i &:= \sqrt{m + 2} \Phi_i^{(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)})) \xi_{n+1}, \\ X_i^\alpha(t) &:= \exp_{X_i^\alpha(t_n^{(\alpha)})}^{(t_n^{(\alpha)})} \left(\frac{t - t_n^{(\alpha)}}{\alpha^2} \left(\alpha \tilde{\xi}_{n+1}^i + \alpha^2 Z(t_n^{(\alpha)}) \right) \right) \end{aligned}$$

for $i = 1, 2$. We can easily verify that X_i^α has the same law as X^α with $x_0 = x_i$.

In what follows, we assume (4.1). We can easily verify that it implies Assumption 1. Thus, by Theorem 3.1, $(\mathbf{X}^\alpha)_{\alpha > 0}$ is tight under Assumption 1. In addition, a subsequential limit $\mathbf{X}^{\alpha_k} \rightarrow \mathbf{X} = (X_1, X_2)$ in law exists and it is a coupling of two \mathcal{L}_t -diffusion processes starting at x_1 and x_2 respectively. We fix such a subsequence $(\alpha_k)_{k \in \mathbb{N}}$. In the rest of this paper, we use the same symbol \mathbf{X}^α for the subsequence \mathbf{X}^{α_k} and the term “ $\alpha \rightarrow 0$ ” always means the subsequential limit “ $\alpha_k \rightarrow 0$ ”. Set $\hat{\sigma}_R^i := \sigma_R(d_{g(\cdot)}(o, X_i^\alpha(\cdot)))$ for $i = 1, 2$. We fix $R > 1$ sufficiently large until the beginning of the proof of Theorem 4.1. Let $M_0 \subset M$ be a relatively compact open set satisfying (2.1) for $2R$ instead of R .

We first show a difference inequality of $d_{g(t)}(\mathbf{X}^\alpha(t))$. To describe it, we will introduce several notations as in the last section. For simplicity, let us denote $\gamma_{X_1^\alpha(t_n^{(\alpha)})X_2^\alpha(t_n^{(\alpha)})}^{(t_n^{(\alpha)})}$ by $\bar{\gamma}_n$. Let us define a vector field V_{n+1} along $\bar{\gamma}_n$ by

$$V_{n+1} := \parallel_{\bar{\gamma}_n}^{(t_n^{(\alpha)})} \left(\tilde{\xi}_{n+1}^1 - \langle \tilde{\xi}_{n+1}^1, \dot{\bar{\gamma}}_n \rangle_{g(t_n^{(\alpha)})} \dot{\bar{\gamma}}_n(0) \right).$$

Take $v \in \mathbb{R}^m$. Let us define λ_{n+1}^* and Λ_{n+1}^* by

$$\begin{aligned}\lambda_{n+1}^* &:= \begin{cases} 2\langle \tilde{\xi}_{n+1}^1, \dot{\gamma}_n \rangle_{g(t_n^{(\alpha)})} & \text{if } (y_1, y_2) \notin D(M), \\ 2\sqrt{m+2}\langle \xi_{n+1}, v \rangle & \text{otherwise,} \end{cases} \\ \Lambda_{n+1}^* &:= \frac{1}{2} \left(\int_0^{d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)}))} (\partial_t g(t_n^{(\alpha)}) + 2(\nabla Z(t_n^{(\alpha)}))^b) (\dot{\gamma}_n(s), \dot{\gamma}_n(s)) ds \right. \\ &\quad \left. + I_{\tilde{\gamma}_n}^{(t_n^{(\alpha)})}(V_{n+1}) \right) 1_{\{\mathbf{X}^\alpha(t_n^{(\alpha)}) \notin D(M)\}}.\end{aligned}$$

For $\delta \geq 0$, let us define $\tau_\delta : \mathcal{C}_1 \rightarrow [T_1, T_2] \cup \{\infty\}$ by $\tau_\delta(w) := \inf \{t \geq T_1 \mid w(t) \leq \delta\}$. We also define $\hat{\tau}_\delta$ by $\hat{\tau}_\delta := \tau_\delta(d_{g(\cdot)}(\mathbf{X}^\alpha(\cdot)))$.

Lemma 4.4 *For $n \in \mathbb{N}_0$ with $n < N^{(\alpha)}$, we have*

$$\begin{aligned}e^{kt_{n+1}^{(\alpha)}/2} d_{g(t_{n+1}^{(\alpha)})}(\mathbf{X}^\alpha(t_{n+1}^{(\alpha)})) &\leq \left(1 + \frac{k}{2}\right) e^{kt_n^{(\alpha)}/2} d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)})) \\ &\quad + e^{kt_n^{(\alpha)}/2} (\alpha \lambda_{n+1}^* + \alpha^2 \Lambda_{n+1}^*) + o(\alpha^2)\end{aligned}\quad (4.2)$$

when $n < \hat{\tau}_\delta \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2$ and α is sufficiently small. Moreover, we can control the error term $o(\alpha^2)$ uniformly in the position of \mathbf{X}^α .

Proof. When $(t_n^{(\alpha)}, \mathbf{X}^\alpha(t_n^{(\alpha)})) \notin \text{Cut}_{\text{ST}}$, (4.2) is just a consequence of the second variational formula for the distance function combined with the index lemma for $I_{\tilde{\gamma}_n}^{(t_n^{(\alpha)})}$. To include the case $(t_n^{(\alpha)}, \mathbf{X}^\alpha(t_n^{(\alpha)})) \in \text{Cut}_{\text{ST}}$ and to obtain a uniform control of $o(\alpha)$, we extend this argument. Let us define H and $p_1, p_2 : H \rightarrow [T_1, T_2] \times \bar{M}_0 \times \bar{M}_0$ by

$$\begin{aligned}H &:= \left\{ (t, x, y, z) \mid \begin{array}{l} t \in [T_1, T_2], x, y, z \in \bar{M}_0, \\ d_{g(t)}(x, y) \geq \delta, \\ d_{g(t)}(x, y) = 2d_{g(t)}(x, z) = 2d_{g(t)}(y, z) \end{array} \right\}, \\ p_1(t, x, y, z) &:= (t, x, z), \\ p_2(t, x, y, z) &:= (t, y, z).\end{aligned}$$

If $\mathbf{q} = (t, x, y, z) \in H$, then $p_1(\mathbf{q}), p_2(\mathbf{q}) \notin \text{Cut}_{\text{ST}}$ since z is on a midpoint of a minimal $g(t)$ -geodesic joining x, y . Since H is compact, $p_1(H)$ and $p_2(H)$ are also compact. Hence there is a constant $\eta > 0$ such that

$$\inf \left\{ |t - t'| + d_{g(t)}(x, x') + d_{g(t)}(y, y') \mid \begin{array}{l} (t, x, y) \in p_1(H) \cup p_2(H), \\ (t', x', y') \in \text{Cut}_{\text{ST}} \end{array} \right\} > \eta.$$

Take $\alpha > 0$ sufficiently small relative to η and δ . Set

$$\begin{aligned}p_n &:= \bar{\gamma}_n \left(\frac{d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)}))}{2} \right), \\ p'_n &:= \exp_{p_n}^{t_n^{(\alpha)}} \left(V_{n+1} \left(\frac{d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)}))}{2} \right) \right).\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)})) &= d_{g(t_n^{(\alpha)})}(X_1^\alpha(t_n^{(\alpha)}), p_n) + d_{g(t_n^{(\alpha)})}(p_n, X_2^\alpha(t_n^{(\alpha)})), \\ d_{g(t_{n+1}^{(\alpha)})}(\mathbf{X}^\alpha(t_{n+1}^{(\alpha)})) &\leq d_{g(t_{n+1}^{(\alpha)})}(X_1^\alpha(t_{n+1}^{(\alpha)}), p'_n) + d_{g(t_{n+1}^{(\alpha)})}(p'_n, X_2^\alpha(t_{n+1}^{(\alpha)})). \end{aligned}$$

Since $(t_n^{(\alpha)}, \mathbf{X}^\alpha(t_n^{(\alpha)}), \bar{\gamma}_n(d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)}))/2)) \in H$, we can apply the second variational formula to each term on the right hand side of the above inequality. Hence we obtain (4.2). For a uniform control of the error term, we remark that $\bar{\gamma}_n$ is included in M_0 and the $g(t_n^{(\alpha)})$ -length of $\bar{\gamma}_n$ is bigger than δ . These facts follows from $n < \hat{\tau}_\delta \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2$ and the choice of M_0 . Thus the every calculation of the second variational formula above is done on a compact subset of $[T_1, T_2] \times M_0 \times M_0$ which is uniformly away from Cut_{ST} . It yields the desired result. \square

Let us define a continuous stochastic process U_a^α on \mathbb{R} starting at a by

$$U_a^\alpha(t) := e^{-kt/2}a + \alpha e^{-kt/2} \left(\sum_{j=1}^n e^{kt_j^{(\alpha)}/2} \lambda_j^* + \frac{t - t_n^{(\alpha)}}{\alpha^2} e^{kt_n^{(\alpha)}/2} \lambda_{n+1}^* \right).$$

We next show the following comparison theorem for the distance process of coupled geodesic random walks.

Lemma 4.5 *For each $\varepsilon > 0$, there exists a family of events $(E_\varepsilon^\alpha)_\alpha$ such that $\mathbb{P}[E_\varepsilon^\alpha]$ converges to 1 as $\alpha \rightarrow 0$ and*

$$d_{g(t)}(\mathbf{X}^\alpha(t)) \leq U_{d_{g(T_1)}(\mathbf{X}^\alpha(T_1))}^\alpha(t) + \varepsilon \quad (4.3)$$

for all $t \in [T_1, T_2 \wedge \hat{\tau}_\delta \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2]$ on E_ε^α for sufficiently small α .

Proof. In a similar way as in the proof of Lemma 3.10, we can complete the proof once we have found E_ε^α on which (4.3) holds when $t = t_n^{(\alpha)} \in [T_1, T_2 \wedge \hat{\tau}_\delta \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2]$. Set $\bar{\Lambda}_{n+1}^* := \mathbb{E}[\Lambda_{n+1}^* | \mathcal{F}_n]$ and $\bar{\Lambda}_0^* := 0$. Then $\sum_{j=1}^n e^{kt_{j-1}^{(\alpha)}/2} (\Lambda_j^* - \bar{\Lambda}_j^*)$ is an \mathcal{F}_n -local martingale. Indeed, Λ_{n+1}^* is bounded if $n < \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2$ and so is $\bar{\Lambda}_{n+1}^*$. Let us define E_ε^α by

$$E_\varepsilon^\alpha := \left\{ \sup_{\substack{N \leq N^{(\alpha)} \\ t_N^{(\alpha)} \leq T_2 \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2}} \sum_{j=1}^{N+1} e^{kt_j^{(\alpha)}/2} (\Lambda_j^* - \bar{\Lambda}_j^*) \leq \frac{\varepsilon}{2\alpha^2} \right\}.$$

In a similar way as in Lemma 3.8 or [16, Lemma 6], we can show $\lim_{\alpha \rightarrow 0} \mathbb{P}[E_\varepsilon^\alpha] = 1$. Since we have $(m+2)\mathbb{E}[\langle \xi_i, e_k \rangle \langle \xi_i, e_l \rangle] = \delta_{kl}$, we obtain

$$\bar{\Lambda}_{n+1}^* \leq -\frac{k}{2} d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)})).$$

Thus an iteration of Lemma 4.4 implies (4.3) on E_ε^α when $t = t_n^{(\alpha)}$. \square

Proof of Theorem 4.1. Take $\varepsilon \in (0, 1)$ arbitrary. Let $R > 1$ be sufficiently large so that

$$\limsup_{\alpha \rightarrow 0} \mathbb{P} [\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 \leq T_2] < \varepsilon.$$

It is possible by Proposition 3.4. Set $a := d_{g(T_1)}(x_1, x_2)$. Take $T \in [T_1, T_2]$ and let $\delta > 0$ be $\delta > 2\varepsilon$. Then Lemma 4.5 yields

$$\begin{aligned} \mathbb{P} [\hat{\tau}_\delta > T] &\leq \mathbb{P} [\{\hat{\tau}_\delta > T\} \cap E_\varepsilon^\alpha \cap \{\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 > T\}] + 2\varepsilon \\ &\leq \mathbb{P} [\tau_{\delta/2}(U_a^\alpha) > T] + 2\varepsilon. \end{aligned}$$

Thus we obtain

$$\mathbb{P} [\hat{\tau}_\delta > T] \leq \mathbb{P} \left[\inf_{t \in [T_1, T]} U_a^\alpha(t) \geq \delta/2 \right]$$

by letting $\varepsilon \downarrow 0$. Note that U_a^α converges in law to U_a as $\alpha \rightarrow 0$. Since

$$\{\mathbf{w} \in C([T_1, T_2] \rightarrow M \times M) \mid \tau_\delta(d_{g(\cdot)}(\mathbf{w}(\cdot))) > T\}$$

is open and $\{w \mid \inf_{t \in [T_1, T_2]} w(t) \geq \delta/2\}$ is closed in $C([0, T] \rightarrow \mathbb{R})$, the Portmanteau theorem yields

$$\begin{aligned} \mathbb{P} \left[\inf_{T_1 \leq t \leq T} d_{g(t)}(\mathbf{X}(t)) > \delta \right] &\leq \liminf_{\alpha \rightarrow 0} \mathbb{P} [\hat{\tau}_\delta > T] \\ &\leq \limsup_{\alpha \rightarrow 0} \mathbb{P} \left[\inf_{t \in [T_1, T]} U_a^\alpha(t) \geq \delta/2 \right] \leq \mathbb{P} \left[\inf_{t \in [T_1, T]} U_a(t) \geq \delta/2 \right]. \end{aligned}$$

Therefore the conclusion follows by letting $\delta \downarrow 0$. \square

We can also construct a coupling by parallel transport by following our manner. In the construction of the coupling by reflection, we used a map $m_{xy}^{(t)}$. By following the same argument after replacing $m_{xy}^{(t)}$ with $\parallel_{\gamma_{xy}}^{(t)}$, we obtain a coupling by parallel transport. The difference of it from the coupling by reflection is the absence of the term corresponding to λ_n^* , which comes from the first variation of arclength. As a result, we can show the following (cf. [16]):

Theorem 4.6 *Assume (4.1). For $x_1, x_2 \in M$, there is a coupling $\mathbf{X}(t) = (X_1(t), X_2(t))$ of two \mathcal{L}_t -diffusion particles starting at x_1 and x_2 at time T_1 respectively such that*

$$d_{g(t)}(\mathbf{X}(t)) \leq e^{-k(t-s)/2} d_{g(s)}(\mathbf{X}(s))$$

for $T_1 \leq s \leq t \leq T_2$ almost surely.

It recovers a part of results studied in [1]. In particular, a contraction type estimate for Wasserstein distances under the heat flow follows.

Proof. Let us construct a coupling by parallel transport of geodesic random walks $\mathbf{X}^\alpha = (X_1^\alpha, X_2^\alpha)$ starting at $(x_1, x_2) \in M \times M$ by following the procedure stated just

before Theorem 4.6. By taking a subsequence, we may assume that \mathbf{X}^α converges in law as $\alpha \rightarrow 0$. We denote the limit by $\mathbf{X} = (X_1, X_2)$. In what follows, we prove

$$\mathbb{P} \left[\sup_{T_1 \leq s \leq t \leq T_2} (e^{kt/2} d_{g(t)}(\mathbf{X}(t)) - e^{ks/2} d_{g(s)}(\mathbf{X}(s))) > \varepsilon \right] = 0$$

for any $\varepsilon > 0$. By virtue of the Portmanteau theorem together with Proposition 3.4, it suffices to show

$$\lim_{\alpha \rightarrow 0} \mathbb{P} \left[\sup_{T_1 \leq s \leq t \leq T_2} (e^{kt/2} d_{g(t)}(\mathbf{X}^\alpha(t)) - e^{ks/2} d_{g(s)}(\mathbf{X}^\alpha(s))) > \varepsilon, \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 = \infty \right] = 0 \quad (4.4)$$

for any $R > 1$. For simplicity of notations, we write $d_n := e^{kt_n^{(\alpha)}/2} d_{g(t_n^{(\alpha)})}(\mathbf{X}^\alpha(t_n^{(\alpha)}))$ in this proof. For $\delta > 0$, let us define a sequence of \mathcal{F}_n -stopping times S_l by $S_0 := 0$ and

$$\begin{aligned} S_{2l+1} &:= \inf \{j \geq S_{2l} \mid d_j \leq \delta\} \wedge N^{(\alpha)}, \\ S_{2l} &:= \inf \{j \geq S_{2l-1} \mid d_j \geq 2\delta\} \wedge N^{(\alpha)}. \end{aligned}$$

Note that $d_{S_{2l-1}} \leq 3\delta$ holds on $\{\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 = \infty\}$ for sufficiently small α . As mentioned just before Theorem 4.6, Lemma 4.4 holds with $\lambda^* = 0$. Moreover, we can obtain the same estimate (4.2) even when $S_{2l-1} \leq n < S_{2l} \wedge \bar{\sigma}_R^1 \wedge \bar{\sigma}_R^2$ for some $l \in \mathbb{N}_0$. In this case, the error term $o(\alpha^2)$ is controlled uniformly also in l . Let us define an event E_δ^α by

$$E_\delta^\alpha := \left\{ \sup_{\substack{n \leq N \leq N^{(\alpha)} \\ t_N^{(\alpha)} \leq T_2 \wedge \hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2}} \sum_{j=n+1}^{N+1} e^{kt_j^{(\alpha)}/2} (\Lambda_j^* - \bar{\Lambda}_j^*) \leq \frac{\delta}{2\alpha^2} \right\}.$$

Then, as in Lemma 3.8 and Lemma 4.5, we can show $\lim_{\alpha \rightarrow 0} \mathbb{P}[E_\delta^\alpha] = 1$. On $E_\delta^\alpha \cap \{\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 = \infty\}$, we have $d_N \leq d_n + \delta$ for $S_{2l-1} \leq n \leq N \leq S_{2l}$ if α is sufficiently small. Moreover, for $n < S_{2l-1} \leq N < S_{2l}$,

$$d_N - d_n \leq (d_N - d_{S_{2l-1}}) + d_{S_{2l-1}} \leq 5\delta.$$

In the case $S_{2l} \leq N < S_{2l+1}$, we obtain $d_N - d_n \leq 2\delta$. Thus $d_N - d_n \leq 5\delta$ holds for all $n < N$ on $E_\delta^\alpha \cap \{\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 = \infty\}$. Take $\delta > 0$ less than $\varepsilon/10$. Then our observations yield (4.4) since $d_{g(t)}(\mathbf{X}^\alpha(t)) - d_{g(\lfloor t \rfloor_\alpha)}(\mathbf{X}(\lfloor t \rfloor_\alpha))$ becomes uniformly small on $\{\hat{\sigma}_R^1 \wedge \hat{\sigma}_R^2 = \infty\}$ as $\alpha \rightarrow 0$. \square

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